Human Capital Misallocation, TFP and Redistributive Policies*

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Abstract

We analyse the impact of income inequality and redistribution on the misallocation of resources and TFP, in economies with financial market imperfections. We calibrate a model based on Benabou’s (2002) model of human capital, but with the addition of physical capital. In the absence of a credit market TFP losses due to misallocation can be significant and are comparable in size to those found in Hsieh and Klenow (2009). However, redistributive policies aimed at reducing these TFP losses have only small positive effects on TFP but large negative effects on per capita output.

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1 Introduction

The importance of total factor productivity (TFP) for variations in per capita income has been recognized in macroeconomics and development economics ever since Solow’s (1957) seminal paper. In the context of the standard neoclassical growth model, while factor inputs can play a role in these variations in the short run, TFP is the only source in the long run. Understanding the determinants of TFP in a consistent way requires a theory of TFP — something Prescott (1998) famously called for.

Recent work, following Restuccia and Rogerson (2008) and Hsieh and Klenow (2009), has examined the role that the misallocation of resources, due to market distortions of some kind, play in determining TFP differences. Hsieh and Klenow (2009), for example, estimated that, if market distortions were somehow reduced down to US levels, TFP would increase in China by 30-50% and in India by 40-60%. These large numbers have attracted considerable attention, and more recent work has focussed attention on particular markets that may be responsible for these figures.\(^1\)

A particular emphasis has been placed on the role that imperfections in financial markets play in misallocation and TFP differences. Buera et al (2011), for example, estimate that financial frictions can reduce TFP by up to 36%. Midrigan and Xu (2014) found that, while financial frictions do affect misallocation, relative to their effects upon firm entry and technological adoption, these effects are, quantitatively, quite small (reducing TFP by approximately 5% in developing countries). Subsequently, using a Lucas (1978) span-of-control model, based on Buera et al (2011) and Bhattacharya et al (2013), Yoon (2016) found that, by encouraging substitution from physical capital into managerial capital, credit market imperfections can actually increase TFP.

In this paper we examine the implications of financial market imperfections along a different margin, namely, the distribution of human capital — and the consequent effects on misallocation and TFP. Following Benabou (2002), we analyse an economy in which credit markets are absent and in which a non-degenerate distribution of income persists in the long-run equilibrium. This income distribution is influenced by, among other things, a redistributive tax-and-transfer scheme operated by the government, with the key policy parameter being the average marginal income tax rate, \(\tau\). As the government increases the value of \(\tau\) it redistributes from richer households to poorer ones.

In this setting, facing the credit constraint, poorer households are restricted when fi-

\(^1\)For a survey, see Restuccia and Rogerson (2013).
nancing the education of their young. With diminishing marginal returns to education, this, alone, generates inefficiencies due to misallocation. The misallocation problem is worsened, though, when (as in Benabou’s model) children are born with different abilities; particularly when poor parents of high-ability children are not able to afford the appropriate level of education for them. Thus an increase in \( \tau \) can, in principle, mitigate this problem by diminishing the effect of the credit constraint. However, as Benabou points out, this type of redistribution also distorts savings decisions. Higher values of \( \tau \) therefore not only reduce misallocation (a positive effect) but also reduce savings (a negative effect). Under reasonable parameter values, steady-state GDP is inverted-U-shaped in \( \tau \) and, in principle, one can find a value of \( \tau \) that maximizes GDP. \(^2\)

Benabou’s (2002) original model has only one accumulable factor: human capital. Consequently, to use this model to account for changes in TFP, quantitatively, certain adjustments need to be made to the model – namely, we need to allow for another productive factor: physical capital. In this paper, following Tang (2008), we introduce physical capital in a way that preserves many of the convenient mathematical properties of Benabou’s model. In particular, closed-form solutions are available for all of the key endogenous variables – making calibration relatively easy, and the explanations for the effects we consider relatively easy to interpret.

In the spirit of Hsieh and Klenow’s (2009) original exercise, we calibrate the model to three economies: the USA, China, and India, generating equilibrium values of TFP for each country. We then compare these values with the values of TFP that would exist in the absence of misallocation. (This exercise is similar to one undertaken by Hsieh and Klenow’s study and summarized in their Table IV.) We find that the TFP gains from eliminating human capital misallocation in this way are smaller than those found by Hsieh and Klenow, but comparable in terms of orders of magnitude. Thus, we find relatively sizeable effects of capital market imperfections on misallocation and TFP, through the channel of the distribution of human capital.

We then consider a different exercise, by exploiting the policy structure embedded in Benabou’s model. We identify the values of \( \tau \) that maximize the TFP values in each of the countries, in an attempt to gauge the extent to which changes to redistributive

\(^2\)Benabou also considered a variety of alternative policy objectives – each of which has its own optimal \( \tau \). He did not, however, consider TFP specifically. See Tang and King (2005) for a re-appraisal of Benabou’s numbers.
policies could increase those TFP values. We find that, in each country, maximizing
the value of TFP would require significantly more progressive tax and transfer policies
(in the USA and India, for example, this would entail more than a doubling the cur-
rent values of $\tau$). However, even these drastic changes to tax and transfer structure
would, according to the model, have only modest effects on the value of TFP. The
most significant effect would be in India, where TFP would only rise, in this case, by
approximately 1.74%. Moreover, making these policy changes would actually depress
steady state output per capita in these countries.

We conclude, therefore, that, although human capital misallocation may be a sig-
nificant contributor to TFP differences, there is no clear role for redistributive policy
to reduce this misallocation and increase TFP.

The remainder of the paper is structured as follows. The model is presented in
Section 2, its equilibrium dynamics are characterized in Section 3, and its steady state
is analysed in Section 4. In Section 5 we present the quantitative analysis, and Section
6 contains some concluding remarks. Appendix provides proofs of all the key lemmas
and propositions in the paper.

2 The Model

The model is a variant of Benabou’s (2002) model, extended to include physical capital,
together with the human capital included in his original model.

2.1 Model Environment

The economy is populated by a continuum of infinitely lived dynasties, indexed by
$i \in [0, 1]$. Each dynasty is made up of a sequence of families consisting of individuals
who live for two periods: first as a child and then as an adult. In each period $t$, the
dynasty is represented by a family of one adult and one child. The adult, in period $t$,
makes all decisions for that period subject to the constraint that she cannot pass on
debt to her child. We call the adult of dynasty $i$ in period $t$ the "dynastical agent $i$
"or simply "agent $i$". The preferences of agent $i$ in period $t$ are given by:

$$\ln U^i_t = E_t \left[ \sum_{n=0}^{\infty} \rho^n \left( \ln c^i_{t+n} - (l^i_{t+n})^\eta \right) \right], \quad (1)$$
where $c^i_t \geq 0$ and $l^i_t \in [0, 1]$ denote, respectively, consumption and labor supply by the adult of the dynasty $i$ in period $t$; $\rho \in (0, 1)$ is the discount factor, and $\eta > 1$.

We assume that each agent can operate the following production technology$^3$, which converts labor $l^i_t$, physical capital $k^i_t$, and human capital $h^i_t$ into output $y^i_t$:

$$y^i_t = (k^i_t)^\lambda (h^i_t)^\mu (l^i_t)^{1-\lambda-\mu}, \quad (2)$$

where $\lambda, \mu \in (0, 1)$, and $\lambda + \mu < 1$.

As in Benabou (2002), the government has a scheme of progressive income taxation and transfers such that the disposable income of a typical agent at a date $t$, denoted $\check{y}^i_t$, satisfies:

$$\check{y}^i_t \equiv (y^i_t)^{1-\tau} (\tilde{y}_t)^\tau. \quad (3)$$

Here, $\tau \in (0, 1)$ measures the average marginal tax rate and is identified as the degree of redistribution or progressivity in fiscal policy. In this scheme, those with incomes higher than $\check{y}_t$ pay net taxes while those with incomes below $\check{y}_t$ receive net transfers. Thus, $\check{y}_t$ represents the break-even level of income with respect to the tax scheme. The government’s balanced-budget constraint for this scheme is:

$$\int_0^1 (y^i_t)^{1-\tau} (\tilde{y}_t)^\tau \, di = y_t, \quad (4)$$

where $y_t \equiv \int_0^1 y^i_t \, di$ denotes per-capita income.

The government also provides education subsidies and bequest subsidies to offset the negative effects of the income tax on output. They finance those subsidies by taxing consumption at a rate $\theta \in (0, 1)$. At each date $t$, the disposable income of agent $i$ must equal the total expenditure on consumption $c^i_t$, private education expenditure $e^i_t$, and the bequest $b^i_t$:

$$\check{y}^i_t = (1 + \theta) c^i_t + e^i_t + b^i_t. \quad (5)$$

The agent receives an education subsidy at a rate $d \in (0, 1)$ per unit of his/her expenditure $e^i_t$, on the child’s education such that, in the following period, the grown-up child’s human capital $h^i_{t+1}$, as a function of her innate ability $\xi^i_{t+1}$, parental human

\[3\] Notice that, as in Benabou’s model, there are no firms in this model, per se. Each agent operates a production technology directly. The output from this process can be used for consumption, bequests, education expenditure, or taxes – as outlined below.
capital $h_{i+1}^t$, and the sum of private and public investment on her education $(1 + d) e_{i}^t$, is given by

\[ h_{i+1}^t = \kappa \xi_{i+1}^t \left( h_{i}^t \right)^{\alpha} \left( (1 + d) e_{i}^t \right)^{\beta}. \]  

(6)

Here, $\kappa > 0$, $\alpha, \beta \in (0, 1)$ are parameters, and the term $\xi_{i+1}^t$ represents $i.i.d.$ idiosyncratic shocks with $\ln \xi_{i}^t \sim N(\varphi, \sigma^2)$, and constants $\varphi$ and $\sigma$. Conceptually, the shocks arise from heterogeneity in innate ability or in the efficiency of human capital usage across different dynasties at time $t$, and over time within a dynasty. The parameter $\alpha$ measures the child’s human capital elasticity with respect to its parent’s human capital. The parameter $\beta$ measures the corresponding elasticity of human capital with regard to education expenditure (which, arguably, is primarily determined by the quality of the education system).

Each child’s physical capital comes from the parent’s bequest. The depreciation rate is assumed to be 100%. That is, capital goods become obsolete at the end of each generation. This implies that, in each period, agents use new tools and machines to produce output by combining them with their own human capital. Parents use the bequest $b_t^i$ to buy new tools for their children at the subsidized rate with subsidy $v \in (0, 1)$ set by the government such that in the period $t + 1$, the agent $i$’s physical capital $k_{i+1}^t$ satisfies:

\[ k_{i+1}^t = (1 + v) b_t^i. \]  

(7)

The government also faces a budget constraint for the consumption taxes, the education subsidies, and the bequest subsidies:

\[ \theta \int_0^1 c_t^i di = d \int_0^1 e_t^i di + \nu \int_0^1 b_t^i di. \]  

(8)

Initial endowments of physical and human capital ($k_0^i$ and $h_0^i$) are jointly lognormally distributed and the adult receives one unit of labor endowment in each period.

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4 As in Lucas (1988) and Glomm and Ravikumar (1992), here, that each agent’s stock of human capital depends on the parent’s stock of human capital, time spent in school, and the quality of schools.

5 In the context of human capital inequality, Benabou (1996) called this a “neighborhood externality”.
2.2 Individual Optimization

For each date $t$, let $m_{ht}$, $m_{kt}$ denote the means and $\Delta^2_{ht}$, $\Delta^2_{kt}$ denote the variances of $\ln h^i_t$ and $\ln k^i_t$, respectively, and let $\text{cov}_t$ denote the covariance between $\ln h^i_t$ and $\ln k^i_t$. Let $M_t \equiv (m_{ht}, m_{kt}, \Delta^2_{ht}, \Delta^2_{kt}, \text{cov}_t)$. The stationary policy sequence is $T \equiv (\tau, d, v, \theta)$. Then for the agent’s dynamic optimization problem, the state variables are $(h^i_t, k^i_t, M_t, T)$, and the control variables are $(l^i_t, e^i_t, b^i_t)$ and the Bellman equation is:

$$\ln U(h^i_t, k^i_t, M_t; T) = \max_{l^i_t, e^i_t, b^i_t} \left\{ (1 - \rho) \left[ \ln c^i_t - (h^i_t)^\eta \right] + \rho E_t \left[ \ln U(h^i_{t+1}, k^i_{t+1}, M_{t+1}; T) \right] \right\},$$

subject to (2), (3), (5), (6) and (7).

As is common in this type of model, the first order conditions associated with the Bellman equation described by (9) yield simple closed-form solutions for each agent’s optimal labor supply, education expenditure rates, and bequest rates. We present each of these in the following two lemmas, starting with labor supply.

**Lemma 1** The optimal labor supply is invariant to time and the individual and aggregate state variables $(h^i_t, k^i_t, M_t)$ and decreases with the average marginal income tax rate $\tau$:

$$l^i_t = l \equiv \frac{((1 - \lambda - \mu) / \eta) (1 - \rho \alpha) (1 - \tau)}{(1 - \rho \alpha) (1 - \rho \lambda (1 - \tau)) - \rho \beta \mu (1 - \tau)} \frac{1}{1/\eta}.$$

(10)

Next we consider investment rates for the two forms of capital. We denote by $s^j_{jt}$, $j = 1, 2$, respectively the fraction of disposable income that agent $i$ invests in her children’s education and for her bequest such that $s^i_t \equiv e^i_t / y^i_t$, $s^i_{2t} \equiv b^i_t / y^i_t$.

**Lemma 2** The optimal education investment rate $s^i_{1t}$ and the bequest rate $s^i_{2t}$ are invariant to time and the individual and aggregate state variables $(h^i_t, k^i_t, M_t)$ and decrease with the average marginal income tax rate $\tau$ such that:

$$s^i_{1t} = s_1 \equiv \frac{\rho \beta \mu (1 - \tau)}{1 - \rho \alpha} \equiv (1 - \tau) \bar{s}_1,$$

(11)

$$s^i_{2t} = s_2 \equiv \rho \lambda (1 - \tau) \equiv (1 - \tau) \bar{s}_2.$$

(12)

where $\bar{s}_1$ and $\bar{s}_2$ denote the laissez-faire saving rates.
Lemmas 1 and 2 make clear the negative effects of redistribution on the incentives to supply labor and capital inputs. Intuitively, \( \tau \) is a distortionary tax, which reduces the return, from each individual’s point of view, from both labor supply and investment. However, as we will see below, and as is standard in this type of model, \( \tau \) can have an offsetting positive effect through the equilibrium dynamics. We now turn to analyse these dynamics.

### 3 The Equilibrium Dynamics

The optimization problem (9) yields (10)–(12) and the following decision rules:

\[
\ln c^i_t = \ln (1 - s_1 - s_2) - \ln (1 + \theta) + (1 - \tau) \ln y^i_t + \tau \ln \bar{y}_t, \tag{13}
\]

\[
\ln e^i_t = \ln s_1 + (1 - \tau) \ln y^i_t + \tau \ln \bar{y}_t, \tag{14}
\]

\[
\ln b^i_t = \ln s_2 + (1 - \tau) \ln y^i_t + \tau \ln \bar{y}_t. \tag{15}
\]

Together with the government’s budget constraint (4), the above decision rules imply a unique sequence of aggregate state variables \( \{M_t\} \) that coincides with what the agent \( i \) takes as given in (9) such that, at each date \( t = 0, 1, 2, \ldots \), the following aggregate consistency condition holds:

\[
\int_0^1 y^i_t \, di = \int_0^1 [c^i_t + e^i_t (1 + d) + b^i_t (1 + v)] \, di. \tag{16}
\]

### 3.1 The Dynamic Paths of Physical Capital, Human Capital and Income

The dynamic path of physical capital for dynasty \( i \) can be found by combining the logarithm of (7) with (2) and (15):

\[
\ln k^i_{t+1} = \ln \bar{s}_2 + (1 - \lambda - \mu) (1 - \tau) \ln l + \lambda (1 - \tau) \ln k^i_t + \mu (1 - \tau) \ln h^i_t + \tau \ln \bar{y}_t. \tag{17}
\]
Similarly, the dynamic path of human capital can be found by combining the logarithm of (6) with (2) and (14):

\[
\ln h_{i+1} = \ln \kappa + \beta \ln s_1 + \beta (1 - \lambda - \mu) (1 - \tau) \ln l + \ln \xi_{t+1} \\
+ \beta \lambda (1 - \tau) \ln k_i^i + (\alpha + \beta \mu (1 - \tau)) \ln h_i^i + \beta \tau \ln \bar{y}_t. 
\]

Also, substitution of (17) and (18) into (2) yields the equilibrium path of income for agent \( i \):

\[
\ln y_{i+1}^i = \psi + (1 - \alpha) (1 - \lambda - \mu) \ln l \\
+ \mu \ln \xi_{t+1}^i + (\lambda + \beta \mu) \tau \ln \bar{y}_t - \alpha \lambda \tau \ln \bar{y}_{t-1} \\
+ (\alpha + (\lambda + \beta \mu) (1 - \tau)) \ln y_t^i - \alpha \lambda (1 - \tau) \ln y_{t-1}^i, 
\]

where \( \psi \equiv \mu \ln \kappa + \mu \beta \ln \bar{s}_1 + \lambda (1 - \alpha) \ln \bar{s}_2 \) and where \( \bar{s}_1 \) and \( \bar{s}_2 \) are given by (11) and (12).

Notice that the intergenerational persistence of human capital \( p^h \equiv \alpha + \beta \mu (1 - \tau) \) and physical capital \( p^k \equiv \lambda (1 - \tau) \) together imply intergenerational persistence of income \( p \equiv \alpha + (\lambda + \beta \mu) (1 - \tau) \) between parents and children. This has a structural component \( \alpha \) — the child’s human capital elasticity with respect to the parent’s human capital, which cannot be lowered with redistribution alone. The other component of intergenerational persistence decreases with \( \tau \), the degree of redistribution, and through this channel a policy of redistribution enhances intergenerational income mobility. Next, we characterize the dynamic paths of the aggregate state variables.

### 3.2 Dynamics of the Economy-wide State Variables

Given the initial lognormal distribution, by (17) and (18), physical and human capital and income are also lognormally distributed over time. Thus, at each date \( t \), \( M_t \) satisfies:

\[
m_{kt+1} = \ln \bar{s}_2 + (1 - \lambda - \mu) \ln l + \lambda m_{kt} + \mu m_{ht} \\
+ \tau (2 - \tau) \left( \lambda^2 \Delta_{kt}^2 + \mu^2 \Delta_{ht}^2 + 2 \lambda \mu \text{cov}_{ht} \right) / 2, \\
\Delta_{kt+1}^2 = (1 - \tau)^2 \left( \lambda^2 \Delta_{kt}^2 + \mu^2 \Delta_{ht}^2 + 2 \lambda \mu \text{cov}_{ht} \right), 
\]

where \( m_{kt} \) and \( m_{ht} \) are given as in (11) and (12).
\[ m_{ht+1} = \ln \kappa + \varphi + \beta \ln \bar{s}_1 + \beta (1 - \lambda - \mu) \ln l \]
\[ + \beta \lambda m_{kt} + (\alpha + \beta \mu) m_{ht} \]
\[ + \beta \tau (2 - \tau) \left( \lambda^2 \Delta^2_{kt} + \mu^2 \Delta^2_{ht} + 2 \lambda \mu \text{cov}_t \right) / 2, \]

\[ \Delta^2_{ht+1} = \sigma^2 + \beta^2 \lambda^2 (1 - \tau)^2 \Delta^2_{kt} + (\alpha + \beta \mu (1 - \tau))^2 \Delta^2_{ht} \]
\[ + 2 \beta \lambda (1 - \tau) (\alpha + \beta \mu (1 - \tau)) \text{cov}_t, \]

\[ \text{cov}_{t+1} = \beta \lambda^2 (1 - \tau)^2 \Delta^2_{kt} + \mu (1 - \tau) (\alpha + \beta \mu (1 - \tau)) \Delta^2_{ht} \]
\[ + \lambda (1 - \tau) (\alpha + 2 \beta \mu (1 - \tau)) \text{cov}_t. \]

We define, for each date \( t \), an index of inequality \( \Lambda_t \) as the logarithm of the ratio of mean to median income.\(^6\) The following Lemma describes the equilibrium dynamics of per capita income and inequality jointly.

**Lemma 3** At each date \( t \), the inequality index \( \Lambda_t \) equals the variance of logarithmic earnings of agents such that

\[ \Lambda_t = (\lambda^2 \Delta^2_{kt} + \mu^2 \Delta^2_{ht} + 2 \lambda \mu \text{cov}_t) / 2, \]  

and the evolution of earnings of adults is governed by a lognormal distribution such that \( \ln y^i_t \sim N(\lambda m_{kt} + \mu m_{ht} + (1 - \lambda - \mu) \ln l, 2 \Lambda_t). \)

Also, the break-even level of income \( \tilde{y}_t \) satisfies:

\[ \ln \tilde{y}_t = \ln y_t + (1 - \tau) \Lambda_t, \]
\[ = \lambda m_{kt} + \mu m_{ht} + (1 - \lambda - \mu) \ln l + (2 - \tau) \Lambda_t. \]

Following Solow (1957), TFP is defined to be the ratio of average output to the weighted average of inputs. We use a Cobb-Douglas production technology similar to

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\(^6\)This is a particularly convenient index of inequality for this class of models. See Benabou (2002) for a discussion.
those assigned to each individual to compute the TFP for the economy as follows

\[
TFP = \frac{\int_0^1 y^i di}{\left(\int_0^1 k^i di\right)\lambda \left(\int_0^1 h^i di\right)^\mu \left(\int_0^1 l^i di\right)^{1-\lambda-\mu}}. \tag{27}
\]

**Lemma 4** In equilibrium, TFP satisfies

\[
TFP_t = \exp \left( ((\lambda - 1)\lambda \Delta_{kt}^2 + (\mu - 1)\mu \Delta_{kt}^2 + 2\lambda \mu \text{cov}_t) / 2 \right). \tag{28}
\]

Notice that TFP, here, is a function of the dispersion of productive assets – as in Restuccia and Rogerson (2008), Hsieh and Klenow (2009) and others.

## 4 The Steady State

The following proposition identifies a necessary and sufficient condition for the existence of a unique, stable, steady state equilibrium.

**Proposition 1:** The equilibrium sequence \( M_t \) monotonically converges to a unique steady state, as a function of \( \tau \), if and only if \((1 - \alpha)(1 - \lambda) > \beta \mu\).

Intuitively, the condition \((1 - \alpha)(1 - \lambda) > \beta \mu\) implies economy-wide diminishing returns to input accumulation.\(^7\)

In this steady state, our index of inequality is time invariant: \( \Lambda_t = \Lambda \). Using (21), (23) and (24) in (25), we obtain:

\[
\Lambda \equiv \frac{\mu^2 (1 + \lambda \alpha (1 - \tau))}{(1 - \lambda \alpha (1 - \tau)) \left((1 + \lambda \alpha (1 - \tau))^2 - ((\lambda + \beta \mu) (1 - \tau) + \alpha)^2\right)} \frac{\sigma^2}{2}. \tag{29}
\]

By (29), \( \partial \Lambda / \partial \tau < 0 \), that is, an increase in \( \tau \) decreases the steady state inequality \( \Lambda \) (as one should expect).

Closed form solutions for the key variables in the steady state equilibrium are provided in the following proposition.

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\(^7\)In the limiting case, when \((1 - \alpha)(1 - \lambda) = \beta \mu\), the economy exhibits balanced endogenous growth. In this paper we focus on income levels and, so, set aside issues of endogenous growth.
Proposition 2: The steady state equilibrium is characterized by the following equations:

\[ y = TFP \ast k^\lambda h^\mu l^{1-\lambda-\mu}, \]  
\[ \ln k = m_k + \Delta_k^2/2, \]  
\[ \ln h = m_h + \Delta_h^2/2, \]  
\[ TFP \equiv \exp \left( \frac{((\lambda - 1) \lambda \Delta_k^2 + (\mu - 1) \mu \Delta_h^2 + 2\lambda \mu \text{cov})}{2} \right), \]  
\[ m_h = \frac{\left( 1 - \lambda \right) (\ln \kappa + \varphi + \beta \ln s_1) + \beta \lambda \ln s_2}{(1 - \lambda) (1 - \lambda - \beta \mu)} + \beta (1 - \lambda - \mu) \ln l + \beta \tau (2 - \tau) \Lambda, \]  
\[ m_k = \frac{\mu (1 - \lambda) (\ln \kappa + \varphi + \beta \ln s_1) + (1 - \lambda) (1 - \alpha - \beta \mu) \ln s_2}{(1 - \lambda) (1 - \alpha) (1 - \lambda - \beta \mu)}, \]  
\[ \Delta_k^2 = \frac{\mu^2 (1 - \tau)^2 (1 + \lambda \alpha (1 - \tau))}{(1 + \lambda \alpha (1 - \tau))((1 + \lambda \alpha (1 - \tau))^2 - ((\lambda + \beta \mu) (1 - \tau) + \alpha)^2)} \sigma^2, \]  
\[ \Delta_h^2 = \frac{(1 - \lambda (1 - \tau)) (\alpha + 2\beta \mu (1 - \tau)) (1 - \lambda^2 (1 - \tau)^2) - 2\mu \beta \lambda^3 (1 - \tau)^4}{(1 - \lambda \alpha (1 - \tau))((1 + \lambda \alpha (1 - \tau))^2 - ((\lambda + \beta \mu) (1 - \tau) + \alpha)^2)} \sigma^2, \]  
\[ \text{cov} = \frac{\beta \lambda^2 \mu^2 (1 - \tau)^4 + \mu (1 - \tau) (\alpha + \beta \mu (1 - \tau)) (1 - \lambda^2 (1 - \tau)^2)}{(1 - \lambda \alpha (1 - \tau))((1 + \lambda \alpha (1 - \tau))^2 - ((\lambda + \beta \mu) (1 - \tau) + \alpha)^2)} \sigma^2, \]

where \( l, s_1, s_2 \) and \( \Lambda \) are given by (10), (11), (12) and (29).

Thus, here, TFP measures the distance between actual production and the production possibility frontier. By (33) and (36)-(38), it can be seen that it is affected by institutional and technological parameters, such as the elasticity of a child’s human capital with respect to the parent’s human capital \( \alpha \), the quality of education system \( \beta \), the shares of physical \( \lambda \) and human \( \mu \) capital, the demographic parameter \( \sigma^2 \) and the policy parameter \( \tau \).
5 Quantitative Analysis

In this section, we first evaluate the key parameter values for the U.S., Indian and Chinese economies, generate equilibrium TFP values for each country, and, thereby, estimate the losses to TFP attributable to human capital misallocation, given current policy settings. We then generate values of the policy parameter $\tau$ that maximize the equilibrium TFP values in each country and estimate the increments to TFP that would be forthcoming if each country chose those $\tau$ values. We also consider the implications, from moving to those $\tau$ values, upon output per capita in the steady state equilibrium. Finally, we present a sensitivity analysis, to assess the quantitative response of TFP to changes in the parameter values.

5.1 Parameter Values

This subsection describes how we calibrated the model. Consider, first, the two exponents on physical and human capital in the production function, respectively: $\lambda$, and $\mu$. For the USA, following Barro, Mankiw, and Sala-i-Martin (1995), and Benabou (2002), we chose $\lambda = 0.3$, $\mu = 0.5$. For China, Baier et al (2006) found that capital’s share in China fluctuated between 46 and 50 percent of GDP between 1978 and 2003. Brandt et al (2008) and Hsieh and Klenow (2009) found similar results. Thus, we chose $\lambda = 0.5$. According to the International Labour Organization, the ratio of the minimum wage to the average wage in China is 0.38. Following Mankiw, Romer and Weil (1992), $\mu = (1 - \text{ratio}) \times \text{labour share}$, where ratio is defined as the ratio of the minimum wage to average wage. Thus, $\mu = (1 - 0.38) \times 0.5 = 0.31$. For India, Hsieh and Klenow (2009) found that the average capital share was 50% over 1994–1995. We therefore chose $\lambda = 0.5$. According to the International Labour Organization, the ratio of minimum wage to average wage is 0.4. Thus, $\mu = (1 - 0.4) \times 0.5 = 0.3$.

For the time discount factor $\rho$: each generation lasts approximately 25 years. With an annual time discount rate around 0.96 in each country, this implies a discount factor of 0.36 for each generation. We chose $\rho = 0.4$ for each country.

For the exponent on labor supply in the utility function $\eta$: following Benabou (2002), we chose $\eta = 6$ for the US. For India, starting with work participation behaviour as reflected in worker population ratio (WPR), for the entire Indian population (all ages), the estimated number of workers in 2004–05 (from the National Sample Survey (NSS)
2004-2005) was 458 million with a WPR of 41.88%. The Chinese workforce stood at 752 million in 2004, with a WPR of 59.10%. The Chinese workforce was thus 60% larger, and its WPR 17 percentage points higher, than that in India. Thus, we chose $l_{\text{India}} = 0.42$ for India and $l_{\text{China}} = 0.6$ for China and used the labour supply equation to solve for $\eta$ for each country using all the other parameters. Thus, we chose $\eta = 8.28$ for China and $\eta = 3.45$ for India.

For the elasticities of human capital with respect to parent human capital and education, $\beta$, we used the following procedure. The intergenerational persistence of income between parents and children is as follows:

$$p(\tau) \equiv \alpha + (\lambda + \beta \mu) (1 - \tau).$$

Setting $\alpha = 0.35$ and $\beta = 0.4$ allows $p(\tau)$ to range from 0.35 to 0.85. This range is consistent with Solon’s (1992) and Mulligan’s (1997) estimation for U.S. intergenerational persistence, 0.3-0.6. For China: Hertz et al (2007) found that the intergenerational schooling correlation around 0.3. Deng et al (2013) estimate and find that the income persistence was in the range 0.47-0.53 and Gong et al (2012) found it to be 0.63. Thus, we chose $\alpha = 0.3$ and $\beta = 0.3$ which allows $p(\tau)$ to range from 0.3 to 0.9. For India: Hnatkovska et al (2013) found income persistence to be in the range 0.55-0.61. Azam and Bhatt (2015) found the intergenerational correlation in education attainment $\beta = 0.5$. Thus, we chose $\alpha = 0.35$ and $\beta = 0.5$ to allow $p(\tau)$ to range from 0.35 to 0.99.

For the variance of innate abilities, $\sigma^2$: following Benabou (2002), we chose $\sigma^2 = 1$ for the US and, thus, the mean of the logarithm of innate ability $\varphi$ is set to -0.5 so that $E[\xi] = 1$. For China: Benabou (2002) measured the family income inequality by using the logarithm of the ratio of mean to median income. From the World Income Inequality Database (WIID), we get the logarithm of the ratio of mean to median income about 0.39 in 2002. Thus, we set $\sigma^2 = 5$, so that the feasible range is [0.26, 0.62]. For India, from the World Income Inequality Database (WIID), we find the logarithm of the ratio of mean to median income about 0.465 in 2004. Thus, we set $\sigma^2 = 4$, so that the feasible range is [0.21, 0.67].

For existing values of the the policy parameter $\tau$, according to OECD Table I.4, $\tau_{\text{U.S.}} = 21.7\%$. From www.tradingeconomics.com, we get $\tau_{\text{India}} = 30\%$ and $\tau_{\text{China}} = 45\%$. 

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The parameter $\kappa$, in the human capital production technology, is chosen so that per capita income in the model matches the corresponding values in the USA, China, and India in 2005.

The following Table summarizes the benchmark parameter values.

<table>
<thead>
<tr>
<th>Targets</th>
<th>Parameter</th>
<th>U.S.</th>
<th>China</th>
<th>India</th>
</tr>
</thead>
<tbody>
<tr>
<td>share of physical capital</td>
<td>$\lambda$</td>
<td>0.3</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>share of human capital</td>
<td>$\mu$</td>
<td>0.5</td>
<td>0.31</td>
<td>0.3</td>
</tr>
<tr>
<td>time discount rate</td>
<td>$\rho$</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>intertemporal elasticity of labor supply,</td>
<td>$\eta$</td>
<td>6</td>
<td>8.28</td>
<td>3.45</td>
</tr>
<tr>
<td>the elasticity of human capital with respect to education,</td>
<td>$\beta$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>the elasticity of human capital wrt parents human capital</td>
<td>$\alpha$</td>
<td>0.35</td>
<td>0.3</td>
<td>0.35</td>
</tr>
<tr>
<td>variance of ability shocks</td>
<td>$\sigma^2$</td>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>redistributive tax and transfer rate</td>
<td>$\tau$</td>
<td>21.7%</td>
<td>45%</td>
<td>30%</td>
</tr>
<tr>
<td>human capital production technology</td>
<td>$\kappa$</td>
<td>2295</td>
<td>87856</td>
<td>7681</td>
</tr>
</tbody>
</table>

### 5.2 Results

In this subsection, we first present results for the benchmark cases and then report some sensitivity analysis.

#### 5.2.1 Benchmark Cases

Table 2 provides information on different values of $\tau$ for each country, and the corresponding values of per capita income and TFP for that country, in the steady state equilibrium. The second column, with the heading $\tau_0$, gives the estimate of the current values of $\tau$, exactly as in Table 1. The third column shows $\tau^*$: the value of $\tau$ that maximizes per capita income. The fourth column gives $\tau^{**}$: the value of $\tau$ that maximizes TFP. Columns five through ten show the corresponding values of output and TFP, under $\tau_0$, $\tau^*$, and $\tau^{**}$ respectively.
The first thing to notice in Table 2 is that, when considering the existing \( \tau_0 \), the Chinese rate is by far the largest, followed by the Indian rate, and the rate in the USA. We can also see that, for the USA, the existing rate \( \tau_0 \) is smaller but quite close to the rate \( \tau^* \) that maximizes its per capita output and far lower than the rate \( \tau^{**} \) that maximizes its TFP. China’s \( \tau_0 \), on the other hand is significantly larger than its \( \tau^* \) but significantly smaller than its \( \tau^{**} \). India’s \( \tau_0 \), like China’s is larger than its \( \tau^* \) but significantly smaller than its \( \tau^{**} \).

In the absence of misallocation, in the model, from equation (33), we can see that TFP is at its maximal value: unity. As in Hsieh and Klenow (2009), for example, TFP gains from eliminating misallocation reflect the elimination of the dispersion of marginal products. We can use unity, therefore, as a frictionless benchmark for TFP, in each country. Table 3 uses this method to identify the percentage gains to TFP from eliminating this dispersion completely. It also uses the values from Table 2 to compute the percentage gains to TFP available by setting the policy rate \( \tau = \tau^{**} \), and the implications for output.

<table>
<thead>
<tr>
<th></th>
<th>( \tau_0 )</th>
<th>( \tau^* )</th>
<th>( \tau^{**} )</th>
<th>( \ln y(\tau_0) )</th>
<th>( \ln y(\tau^*) )</th>
<th>( \ln y(\tau^{**}) )</th>
<th>TFP(( \tau_0 ))</th>
<th>TFP(( \tau^* ))</th>
<th>TFP(( \tau^{**} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>0.217</td>
<td>0.266</td>
<td>0.816</td>
<td>10.794</td>
<td>10.795</td>
<td>10.664</td>
<td>0.857</td>
<td>0.859</td>
<td>0.868</td>
</tr>
<tr>
<td>China</td>
<td>0.450</td>
<td>0.360</td>
<td>0.596</td>
<td>7.915</td>
<td>7.921</td>
<td>7.887</td>
<td>0.560</td>
<td>0.558</td>
<td>0.561</td>
</tr>
<tr>
<td>India</td>
<td>0.300</td>
<td>0.223</td>
<td>0.788</td>
<td>6.889</td>
<td>6.898</td>
<td>6.534</td>
<td>0.610</td>
<td>0.605</td>
<td>0.621</td>
</tr>
</tbody>
</table>

Table 3: TFP Gains and Output Losses

<table>
<thead>
<tr>
<th></th>
<th>H&amp;K (2009, Table IV)</th>
<th>TFP gains without frictions</th>
<th>TFP gains if ( \tau = \tau^{**} )</th>
<th>Output losses if ( \tau = \tau^{**} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S.</td>
<td>42.9%</td>
<td>16.69%</td>
<td>1.26%</td>
<td>-12.18%</td>
</tr>
<tr>
<td>China</td>
<td>86.6%</td>
<td>78.57%</td>
<td>0.20%</td>
<td>-2.73%</td>
</tr>
<tr>
<td>India</td>
<td>127.5%</td>
<td>63.93%</td>
<td>1.74%</td>
<td>-29.88%</td>
</tr>
</tbody>
</table>

The second column in Table 3 reproduces the numbers from Table IV in Hsieh and Klenow (2009), for the year 2005. The third column presents the percentage gains to TFP, in each country, according to our model, from completely eliminating the distortions to TFP due to the misallocation of physical and human capital. It is clear
that these are relatively substantial, and in the same order of magnitude as the gains found in the Hsieh and Klenow study, though somewhat smaller overall – particularly for the US and India.

The right hand column in Table 3 uses Table 2 to compute the percentage gains to TFP that, according to the model, would be achievable if the governments in the countries set their $\tau$ values equal to the values that maximize their respective TFP scores: $\tau^{**}$. These are quite small numbers. According to this model, then, the maximal gains from adjusting $\tau$ in this way would be in India, where TFP would rise by only 1.74%. Moreover, from Table 2, we can see that, for each country, if they did choose to set $\tau = \tau^{**}$ this would actually reduce per capita output – by 12.18%, 2.73%, and 29.88% in the USA, China, and India, respectively.

5.2.2 Sensitivity Analysis

By equations (33) and (36)-(38) we can identify that a number of parameters, such as the elasticities of human capital with respect to parent human capital $\alpha$ and education, $\beta$, and share of physical $\lambda$ and human $\mu$ capital, and the fiscal policy parameter $\tau$ affect TFP. In this section, we plot graphs of TFP against these parameters and discuss how and why TFP varies with them. While varying each parameter, in turn, other parameters are kept at the benchmark values for the USA.  

---

8Using the benchmark parameter values from China or India yields similar results. To save space, we don’t present them here. They are available upon request.
Varying $\alpha$

Figure 1—TFP vs Intergenerational Elasticity of Human Capital, $\alpha$, where $\lambda = 0.3, \mu = 0.5, \beta = 0.4, \sigma^2 = 1, \tau = 21.7\%$.

Figure 1 illustrates the effect of changes in $\alpha$, the elasticity of human capital with respect to parent human capital, on TFP in the steady state equilibrium. It is clear that larger values of $\alpha$ induce smaller values of TFP.\(^9\) Intuitively, as $\alpha$ increases, the influence of a parent’s human capital level on the child’s becomes stronger. The likelihood of a parent with a relatively low human capital level passing on a low level of human capital to the child increases as $\alpha$ increases, for example. Ceteris paribus, this will increase disparities in the steady state, which reduce TFP. Recall that, in the benchmark parameterization, $\alpha = 0.35$ in the USA and India, and 0.3 in China. In the neighborhood of these values, any change in $\alpha$ has only small effects on TFP.

\(^9\)Englander and Mittelstadt (1988) find a similar conclusion, empirically.
Varying $\beta$

![TFP vs Quality of Education System](image)

Figure 2—TFP vs Quality of Education System, $\beta$, where $\lambda = 0.3$, $\mu = 0.5$, $\alpha = 0.35$, $\sigma^2 = 1$, $\tau = 21.7\%$.

Figure 2, which shows the response of TFP to the quality of the education system $\beta$, looks similar to Figure 1. Once again, steady state TFP declines as this parameter increases in value. This result is quite different from the standard result that a higher quality of education leads to higher TFP and needs some explanation. The reasoning behind this result is as follows. In the absence of a credit market, children born in rich families can go to school more than those born in poor families – and, thereby, learn more knowledge and skills than children born in poor families. The more important that the educational system is (i.e., the larger $\beta$ is) the larger the knowledge and communication gaps and the higher the levels of income inequality. Consequently, total factor productivity decreases with $\beta$.\(^\text{10}\) In the benchmark parameterization, $\beta = 0.4$ in the USA, 0.3 in China, and 0.5 in India. In the neighborhood of these values, any change in $\beta$ has relatively small effects on TFP.

\(^\text{10}\)This has the implication that, when the credit market is absent or incomplete, reducing barriers to education for the public is important especially when the quality of education is very high.
Varying $\lambda$ and $\mu$

By increasing $\lambda$ and $\mu$ proportionately while keeping the ratio of the two capital shares constant, as implied by Barro, Mankiw and Sala-i-Martin (1995), Figure 3 shows that TFP decreases initially and then starts to increase when $\lambda$ is greater than approximately 0.3. This implies that as an economy develops from an earlier stage of development in which unskilled labours are the main contributors to production to a latter stage in which the economy relies more on skilled labour and tools and machines, TFP decreases. This occurs, again, because of the absence of a credit market. With small values of $\lambda$ and $\mu$, only a small fraction of agents can afford high levels of physical and human capital, and so inequality and dispersion are relatively small and TFP is high. As $\lambda$ and $\mu$ increase, this small fraction increases — which increases inequality and dispersion, reducing TFP. If $\lambda$ and $\mu$ are high enough, though, then most agents can afford high levels of physical and human capital, and so inequality and dispersion fall, and TFP increases. Once again, though, over the range considered, the overall effects on TFP are quite modest.
Figure 4 shows that with higher degrees of redistribution, the level of TFP rises, up to a certain point, then falls. That is, as we saw in Table 2, there exist values of \( \tau \) (denoted \( \tau^{**} \) in that Table) that maximize TFP. Intuitively, for higher values of \( \tau \) income inequality falls (unambiguously), which reduces the variances of physical and human capital. However, high levels of \( \tau \) also imply low levels of the covariance between physical and human capital – which is also an argument in TFP (see equation (33)). When \( \tau \) becomes high enough, this effect overwhelms the effects on the variances, and TFP falls. Once again, though, these effects on TFP, overall, are quite small.

6 Concluding Remarks

We have used a model of income, physical capital, and human capital dispersion – by extending Benabou’s (2002) model to allow for physical capital – to try to quantify the misallocation associated with physical and human capital due to credit market imperfections, and the potential gains to TFP from eliminating this misallocation. This particular model completely shuts down credit markets – clearly an unrealistic
assumption, particularly in the USA – and, so, these numbers should be seen as upper bounds for these gains.

We have found that, while the TFP gains from completely eliminating the misallocation may be rather significant (and comparable in size to those found in Hsieh and Klenow (2009)) the potential gains from using the redistributive tax and transfer policy instrument in Benabou’s model are very modest, and may lead to quite significant reductions in per capita income. According to our model, this potential reduction in per capita income is particularly acute in India, which stands to lose roughly 30% of existing income if this policy were enacted.

We conclude, therefore, that, while the loss in TFP from the misallocation of physical and human capital due to income inequality may be sizeable, redistributional policies aimed at reducing this loss should be considered only with significant caution.

Appendix: Proofs of Lemmas and Propositions

PROOFS OF LEMMAS 1 AND 2:

By (5), we rewrite (9) as follows:

\[ \ln U (h_t^i, k_t^i, M_t; T) = \max_{s_{1t}, s_{2t}} \left\{ (1 - \rho) \left[ \ln (1 - s_{1t}) - \ln (1 + \theta) \right] + \ln \left[ \frac{\ln \gamma_t - (l_t^i)^\eta}{(1 + d)} \right] \right\}. \]

Each agent solves (40) subject to (2), (3) and

\[ h_{t+1}^i = \kappa \left( (1 + d) s_{1t} \right)^{\beta} \left( k_t^i \right)^{\alpha+\beta \mu(1-\tau)} \left( l_t^i \right)^{\beta(1-\lambda-\mu)(1-\tau)} \left( \bar{y}_t \right)^{\beta \tau}, \] and

\[ k_{t+1}^i = (1 + v) s_{2t} \left( k_t^i \right)^{\lambda(1-\tau)} \left( h_t^i \right)^{\mu(1-\tau)} \left( l_t^i \right)^{\beta(1-\lambda-\mu)(1-\tau)} \left( \bar{y}_t \right)^{\beta \tau}. \]

We guess the value function as: \( \ln U (h_t^i, k_t^i, M_t; T) = Z_1 \ln h_t^i + Z_2 \ln k_t^i + B_t. \) Then by
substituting this value function, (41) and (42) into (40), we get

\[
\begin{align*}
Z_1 \ln h^i_t + Z_2 \ln k^i_t + B_t &= (1 - \rho) \left( \frac{\ln (1 - s^i_{1t} - s^i_{2t}) - \ln (1 + \theta)}{\ln l^i_t + \tau \ln \bar{y}_t - (l^i_{1t})^n} ight) \\
&\quad + (1 - \rho + \rho \beta Z_1 + \rho Z_2) \lambda (1 - \tau) \ln k^i_t \\
&\quad + ((1 - \rho + \rho \beta Z_1 + \rho Z_2) \mu (1 - \tau) + \rho \alpha Z_1) \ln h^i_t \\
&\quad + \rho \left( Z_1 \left( \frac{\ln \kappa + \beta \ln (s^i_{1t} (1 + d)) + \varphi}{\ln (s^i_{2t} (1 + v)) + \tau \ln \bar{y}_t} \right) \\
&\quad + Z_2 \left( \ln (s^i_{2t} (1 + v)) + \tau \ln \bar{y}_t \right) \right) \\
&\quad + B_{t+1} \\
\end{align*}
\]

(43)

Taking partial derivatives with respect to \( \ln k^i_t \) and \( \ln h^i_t \) yields:

\[
\begin{align*}
Z_1 &= (1 - \rho + \rho \beta Z_1 + \rho Z_2) \mu (1 - \tau) + \rho \alpha Z_1, \\
(44) \\
Z_2 &= (1 - \rho + \rho \beta Z_1 + \rho Z_2) \lambda (1 - \tau). \\
(45)
\end{align*}
\]

Rearranging (44) and (45), we verify the guess and confirm the existence of (43) and get

\[
\begin{align*}
Z_1 &= \frac{(1 - \rho) \mu (1 - \tau)}{(1 - \rho \alpha) (1 - \rho \lambda (1 - \tau)) - \rho \beta \mu (1 - \tau)}, \\
(46) \\
Z_2 &= \frac{(1 - \rho \alpha) (1 - \rho) \lambda (1 - \tau)}{(1 - \rho \alpha) (1 - \rho \lambda (1 - \tau)) - \rho \beta \mu (1 - \tau)}. \\
(47)
\end{align*}
\]

The values of human capital and physical capital as expressed by their utility elasticities are respectively given by \( Z_1 \) and \( Z_2 \). Note the tax rate \( \tau \) can alter these values individually but does not alter the relative value of human to physical capital, \( \frac{\mu}{\lambda (1 - \rho \alpha)} \), which increases with the output elasticity of human capital \( \mu \), the neighborhood effect \( \alpha \) and patience \( \rho \) but remains unaffected by the quality of education \( \beta \).

The first-order conditions of (40) with respect to the saving rates and labour supply are

\[
\begin{align*}
\frac{1 - \rho}{1 - s^i_{1t} - s^i_{2t}} &= \rho \left( \frac{\partial \ln U^i_{t+1}}{\partial \ln h^i_{t+1}} \frac{\partial \ln h^i_{t+1}}{\partial s^i_{1t}} + \frac{\partial \ln U^i_{t+1}}{\partial \ln k^i_{t+1}} \frac{\partial \ln k^i_{t+1}}{\partial s^i_{1t}} \right), \\
(48) \\
\frac{1 - \rho}{1 - s^i_{1t} - s^i_{2t}} &= \rho \left( \frac{\partial \ln U^i_{t+1}}{\partial \ln h^i_{t+1}} \frac{\partial \ln h^i_{t+1}}{\partial s^i_{2t}} + \frac{\partial \ln U^i_{t+1}}{\partial \ln k^i_{t+1}} \frac{\partial \ln k^i_{t+1}}{\partial s^i_{2t}} \right), \\
(49)
\end{align*}
\]

23
\[(1 - \rho) \eta (l^i_t)^{\eta - 1} = (1 - \rho) (1 - \lambda - \mu) (1 - \tau) / l^i_t \]
\[+ \rho \left( \frac{\partial \ln U^i_{t+1}}{\partial \ln h^i_{t+1}} \frac{\partial \ln h^i_{t+1}}{\partial l^i_t} + \frac{\partial \ln U^i_{t+1}}{\partial \ln k^i_{t+1}} \frac{\partial \ln k^i_{t+1}}{\partial l^i_t} \right) \]

where \(\partial \ln k^i_{t+1}/\partial s^i_{1t} = 0, \partial \ln k^i_{t+1}/\partial s^i_{2t} = 1/s^i_{2t}, \partial \ln h^i_{t+1}/\partial s^i_{1t} = \beta/s^i_{1t}, \partial \ln h^i_{t+1}/\partial s^i_{2t} = 0, \partial \ln k^i_{t+1}/\partial l^i_t = (1 - \lambda - \mu) (1 - \tau) / l^i_t \) and \(\partial \ln h^i_{t+1}/\partial l^i_t = \beta (1 - \lambda - \mu) (1 - \tau) / l^i_t \).

The above optimization problem (43) is strictly concave. Consequently, (48)-(50) are sufficient for the optimization exercise and the Lemmas 1 and 2 follow immediately after we substitute (46) and (47) into (48)-(50).

In order to offset some of the distortionary effects of income taxes on savings for education expenditure and bequest, the government chooses the subsidy rates \(d\) and \(v\) such that:

\[(1 + d) s_1 = \bar{s}_1, \quad (51)\]
\[(1 + v) s_2 = \bar{s}_2. \quad (52)\]

It finances those subsidies with consumption tax by setting the tax rate \(\theta\). Thus, by (8) and Lemma 2, it follows that

\[\frac{\theta (1 - s_1 - s_2)}{1 + \theta} = ds_1 + vs_2, \quad (53)\]

and by (51), (52) and (53), the subsidy rates \(d\) and \(v\) and the consumption tax rate \(\theta\) satisfy,

\[d = \frac{\tau}{1 - \tau}, \quad v = \frac{\tau}{1 - \tau} \text{ and } \theta = \frac{\bar{s}_1 + \bar{s}_2}{1 - \bar{s}_1 - \bar{s}_2 \tau}. \quad (54)\]

The above equation shows, in the equilibrium, that \((d, v, \theta)\) are functions of \(\tau\). It implies that the government could decide the degree of redistribution simply by changing \(\tau\) only. Therefore, we refer to \(\tau\) as the degree of redistribution rather than just the average marginal tax rate. When the government chooses to subsidize education only or not to subsidize at all, \(v\) or both \(d\) and \(v\) will be set to zero by adjusting \(\theta\) accordingly in equilibrium. □

**PROOF OF LEMMA 3:** By assumption, at the initial date \(t = 0\), physical and human capitals are lognormally distributed. By (17) and (18), it follows that \(k^i_t\) and \(h^i_t\) remain lognormally distributed over time and hence by (2) \(y^i_t\) is lognormal and is
given by,
\[ \ln y^i_t = \lambda \ln k^i_t + \mu \ln h^i_t + (1 - \lambda - \mu) \ln l. \tag{55} \]

The mean of the lognormal distribution of \( y^i_t \) is given by,
\[ \int_0^1 \ln y^i_t \, di = \lambda m_{kt} + \mu m_{ht} + (1 - \lambda - \mu) \ln l. \tag{56} \]

The variance of \( \ln y^i_t \) is the sum of variances of \( \ln k^i_t \), \( \ln h^i_t \) plus the covariance of these two variables
\[ \text{var} [\ln y^i_t] = \lambda^2 \Delta^2_{kt} + \mu^2 \Delta^2_{ht} + 2 \lambda \mu \text{cov}_{kht}. \tag{57} \]

The income per capita \( y_t \) is
\[ y_t = \int_0^1 y^i_t \, di = \exp \left( \int_0^1 \ln y^i_t \, di + \frac{1}{2} \text{var} [\ln y^i_t] \right). \tag{58} \]

The median income is
\[ y_{t, median} = \exp \left( \int_0^1 \ln y^i_t \, di \right). \tag{59} \]

Therefore, the inequality index is
\[ \Lambda_t \equiv \log \left( \frac{y_t}{y_{t, median}} \right) = \frac{1}{2} \text{var} [\ln y^i_t] = \left( \lambda^2 \Delta^2_{kt} + \mu^2 \Delta^2_{ht} + 2 \lambda \mu \text{cov}_{kht} \right) / 2. \tag{60} \]

To derive the expression for the break-even point defined by (4), we note the mean of \( y^i_t \) in logarithms, by (58), satisfies
\[ \ln y_t = \lambda m_{kt} + \mu m_{ht} + (1 - \lambda - \mu) \ln l + \Lambda_t, \tag{61} \]
and the mean of \((y^i_t)^{1-\tau}\) in logarithms is
\[ \ln \int_0^1 \frac{L \ln y^i_t}{L} \, di = (1 - \tau) (\lambda m_{kt} + \mu m_{ht} + (1 - \lambda - \mu) \ln l) + (1 - \tau)^2 \Lambda_t. \tag{62} \]

Taking the difference between before and after tax income yields
\[ \ln y_t - \ln \int_0^1 \frac{L \ln y^i_t}{L} \, di = \lambda \tau m_{kt} + \mu \tau m_{ht} + (1 - \lambda - \mu) \tau \ln l + \tau (2 - \tau) \Lambda_t. \tag{63} \]
This implies that $\tau \ln \tilde{y}_t = \lambda \tau m_{kt} + \mu \tau m_{ht} + (1 - \lambda - \mu) \tau \ln l + \tau (2 - \tau) \Lambda_t$, and we get (26). □

PROOF OF LEMMA 4: By (17) and (18), we know that physical and human capital distribute lognormally. Then from the property of the moment generating function for lognormal distribution, we get

$$\ln k = m_k + \Delta^2_k / 2 \text{ and } \ln h = m_h + \Delta^2_h / 2.$$  \hspace{1cm} (64)

Substituting (64) into (61) yields

$$y = k^\lambda h^\mu l^{1-\lambda-\mu} \exp \left( ((\lambda - 1) \lambda \Delta^2_k + (\mu - 1) \mu \Delta^2_h + 2\lambda\mu \text{cov}) / 2 \right).$$  \hspace{1cm} (65)

Then, by (27), we can get (28). □

PROOF OF PROPOSITION 1: Writing the system of linear equations (21), (23) and (24) in a matrix form, we get

$$M_{t+1} = A_0 + A_1 \cdot M_t,$$  \hspace{1cm} (66)

where

$$M_{t+1} \equiv \begin{bmatrix} \Delta^2_{kt+1} \\ \Delta^2_{ht+1} \\ \text{cov}_{t+1} \end{bmatrix}, \quad A_0 \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$A_1 \equiv \begin{bmatrix} (1 - \tau)^2 \lambda^2 & (1 - \tau)^2 \mu^2 & 2\lambda\mu (1 - \tau)^2 \\ (1 - \tau)^2 \beta^2 \lambda^2 & (\alpha + \beta\mu (1 - \tau))^2 & 2\beta\lambda (1 - \tau) (\alpha + \beta\mu (1 - \tau)) \\ (1 - \tau)^2 \beta^2 \lambda^2 & \mu (1 - \tau) (\alpha + \beta\mu (1 - \tau)) & \lambda (1 - \tau) (\alpha + 2\beta\mu (1 - \tau)) \end{bmatrix}.$$  \hspace{1cm} (66)

The sequence $M_t$ converges to a steady state if all eigenvalues of $A_1$, denoted by $S(A_1) \equiv (E_j), j = 1, 2, 3$, are less than one\(^\text{11}\). By setting $\tau = 0$ to avoid unnecessary details, we solve $\text{det} | A_1 - S(A_1) \cdot I | = 0$, where $I$ is identity matrix, to get

$$E_1 \equiv \lambda\alpha < 1,$$  \hspace{1cm} (67)

Eigenvalue $E_1$ is less than one since $\alpha, \lambda \in (0, 1)$. Note that when $\tau = 0$, $|A_1| = \lambda^3 \alpha^3$.

\(^{11}\)For detailed discussion about this property, please see Reich (1949), Lorenz (1993) and Young (2003).
It implies

\[ E_2 \times E_3 = \lambda^2 \alpha^2, \]  

(68)
since \( A_1 \) is symmetric. The symmetry of \( A_1 \) implies also that the trace \( \{ A_1 \} = \sum_{j=1}^{3} E_j \).

Or, equivalently,

\[ E_2 + E_3 = \lambda (\lambda + 2 \beta \mu) + (\alpha + \beta \mu)^2. \]  

(69)

By (68) and (69) and the assumption \((1 - \alpha) (1 - \lambda) - \beta \mu > 0\), it follows that both \( E_2 < 1 \) and \( E_3 < 1 \). Thus, \( S(A_1) < 1 \). Consequently, \( M_t \) converges to a unique steady state which we denote as \( M \). By (66), \( M \) satisfies the following fixed point problem

\[ M = A_0 + A_1 \times M, \]  

(70)

and has a unique solution, since \( I - A_1 \) is nonsingular. It follows, therefore, a unique steady state exists and the equilibrium sequence of \( M_t \) converges to it. Moreover, \( 0 < S(A_1) < 1 \) implies that \( \{ M_t \} \) constitutes a monotone sequence\(^\text{12}\).

Similarly, we can write equations (20) and (22) in a matrix form as below,

\[ N_{t+1} = A_0 + A_1 \times N_t, \]  

(71)

where\(^\text{13}\)

\[ A_0 \equiv \begin{bmatrix} m_{kt+1} \\ m_{ht+1} \end{bmatrix}, \]

\[ A_1 \equiv \begin{bmatrix} \ln \bar{s}_2 + (1 - \lambda - \mu) \ln l + \tau (2 - \tau) \Lambda, \\ \ln \kappa + \varphi + \beta \ln \bar{s}_1 + \beta (1 - \lambda - \mu) \ln l + \beta \tau (2 - \tau) \Lambda \end{bmatrix}. \]

The sequence \( N_t \) converges to a steady state if all eigenvalues of \( A_1 \), denoted by \( S(A_1) \equiv (E_j), j = 1, 2 \), are less than one. We solve \( \det |A_1 - S(A_1) I| = 0 \), where \( I \) is identity

\(^{12}\)For details, see page 255, Lorenz (1993).

\(^{13}\)Note that for simplicity, \( \Delta^2_{kt}, \Delta^2_{ht} \) and \( cov_t \) are regarded as constant here since they converge to steady state in the long run.
matrix, to get
\[
E_1 = \frac{1}{2} (\alpha + \lambda + \beta \mu) + \frac{1}{2} \sqrt{(\alpha + \beta \mu)^2 + 2\beta \lambda \mu - 2\alpha \lambda + \lambda^2}, \\
E_2 = \frac{1}{2} (\alpha + \lambda + \beta \mu) - \frac{1}{2} \sqrt{(\alpha + \beta \mu)^2 + 2\beta \lambda \mu - 2\alpha \lambda + \lambda^2}.
\]

Note that $|A_1| = \alpha \lambda$. It implies that
\[
E_1 \ast E_2 = \alpha \lambda,
\]
since $A_1$ is symmetric. The symmetry of $A_1$ implies also that the trace $\{A_1\} = E_1 + E_2$. Or, equivalently,
\[
E_1 + E_2 = \alpha + \lambda + \beta \mu.
\]
By (73) and (74) and the assumption $(1 - \alpha)(1 - \lambda) - \beta \mu > 0$ and $\alpha, \beta, \lambda, \mu \in (0, 1)$, it follows that both $E_1 < 1$ and $E_2 < 1$. Then a unique steady state exists and the equilibrium sequence of $N_t$ converges to it. The Proposition 1 is proved. □

PROOF OF PROPOSITION 2: From Proposition 1, we know that $\Delta_{kt+1}^2 = \Delta_{kt}^2$, $\Delta_{ht+1}^2 = \Delta_{ht}^2$, $cov_{kt+1} = cov_{t} = cov$, $m_{kt+1} = m_{kt} = m_k$ and $m_{ht+1} = m_{ht} = m_h$. By (21), (23) and (24), we can get (36), (37) and (38). And by (20) and (22), we can get (34) and (35). The derivations of (30) and (33) are from Lemma 4. □

References


