A Note on Indirect Certainty Equivalents for the Firm Facing Price and Production Uncertainty

Robert G. Chambers\textsuperscript{1} and John Quiggin\textsuperscript{2}

August 17, 2001

\textsuperscript{1}Professor, Department of Agricultural and Resource Economics, University of Maryland, College Park and Adjunct Professor, Faculty of Agriculture, University of Western Australia

\textsuperscript{2}Australian Research Council Fellow at the Faculty of Economics, Australian National University, Canberra, ACT
Since its introduction, the Sandmovian model of a risk-averse firm maximizing the expected utility of net returns has been the focus of most economic theorizing on production decisions under price or production uncertainty (Sandmo, 1971). However, relatively little attention has been devoted to characterizing the indirect objective functions, and their associated economic implications, for the various versions of the Sandmovian model. Instead, a folk wisdom seems to have emerged suggesting that little, if any, economic information can be inferred from such indirect objective functions.

Recently, however, Chambers and Quiggin (2000) have clarified the duality between cost functions and their underlying stochastic technologies by showing that standard duality arguments (Shephard, 1970; Färe, 1988; Färe and Primont, 1995) link state-contingent technologies to their dual cost functions. This note shows that it is similarly straightforward to characterize producer decisionmaking under uncertainty by using indirect objective functions. The characterization is for the class of producers with continuous and nondecreasing preferences over stochastic incomes who face both price and production uncertainty. Our general results are independent of both risk preferences and any notion of probability. However, imposing such additional structure on behavior permits one to refine results.

In what follows, we first introduce our notation and model. Then we specify the indirect objective function, and develop its properties for general preferences over stochastic incomes. Next we consider the consequences of particular structural restrictions, such as constant absolute, constant relative risk aversion, constant risk aversion, and several different characterizations of risk aversion for the indirect certainty equivalent. The final section closes.

1 Model and Notation

We are interested in a multiple-output firm facing a stochastic technology. Uncertainty is modelled by ‘Nature’ making a choice from a finite set of states $\Omega = \{1, 2, ..., S\}$. The state-contingent production technology, following Chambers and Quiggin (2000), is modelled by a continuous input correspondence, $X : \mathbb{R}^{M \times S}_+ \to \mathbb{R}^N_+$, which maps matrices of state-
contingent outputs, \( z \), into inputs capable of producing them

\[
X(z) = \{ x \in \mathbb{R}_+^N : x \text{ can produce } z \} \quad z \in \mathbb{R}_+^{M \times S}.
\]

The vector \( z_s \in \mathbb{R}_+^M \) denotes the vector of \textit{ex post} or realized outputs in state \( s \). In addition to continuity, the input correspondence satisfies:\footnote{These properties are discussed in detail in Chambers and Quiggin (2000, Chapter 2). Note, in particular, that they correspond to standard properties placed on input correspondences associated with nonstochastic technologies (Färe, 1988).}

X.1 \( X(0_{M \times S}) = \mathbb{R}_+^N \), and \( O_N \notin X(z) \) for \( z \geq 0_{M \times S} \) and \( z \neq 0_{M \times S} \).

X.2 \( z' \geq z \Rightarrow X(z) \subseteq X(z') \).

X.3 if \( \| \text{vec } z^k \| \rightarrow \infty \) as \( k \rightarrow \infty \), then \( \cap_{k \rightarrow \infty} X(z^k) = \emptyset \).

X.4 \( \lambda X(z_0) + (1 - \lambda) X(z_1) \subseteq X(\lambda z_0 + (1 - \lambda) z_1) \quad 0 < \lambda < 1 \),

Individual producers face stochastic output prices, \( p \in \mathbb{R}_+^{M \times S} \), and non-stochastic input prices, \( w \in \mathbb{R}_+^N \). Their preferences are defined over \textit{ex post} income, \( y \in \mathbb{R}^S \), which is the sum of their holding of a portfolio of financial assets with state-contingent returns \( q \in \mathbb{R}^S \) and their flow profit from production. Hence, their returns in state \( s \) are

\[
y_s = q_s + p_s z_s - w x.
\]

Their evaluation of these \textit{ex post} incomes are given by a continuous and nondecreasing certainty equivalent function, \( e : \mathbb{R}^S \rightarrow \mathbb{R} \) with the property that

\[
e(\mu 1) = \mu, \mu \in \mathbb{R},
\]

where \( 1 \) is the \( S \)-dimensional unit vector.

Following Quiggin and Chambers (1998), \( e \) is said to exhibit constant absolute risk aversion if

\[
e(y + \delta 1) = e(y) + \delta, \quad \delta \in \mathbb{R}.
\]

Preferences exhibit constant relative risk aversion if

\[
e(\mu y) = \mu e(y) \quad \mu > 0, \quad y \in \mathbb{R}_+^S,
\]
and constant risk aversion (Safra and Segal, 1998) if they satisfy both constant relative risk aversion and constant absolute risk aversion, i.e.,

\[ e(\mu (y + \delta 1)) = \mu e(y) + \mu \delta, \quad \mu > 0, \delta \in \mathbb{R}. \]

Although we have defined notions of constant absolute and constant relative risk aversion, we have not yet defined a concept of risk aversion. Indeed, our definitions of constant absolute risk aversion, constant relative risk aversion, and constant risk aversion, despite their nomenclature, are independent of any notion of probability distribution. For continuous certainty equivalents several alternative definitions of aversion to risk are available. For example, Quiggin and Chambers (1998) define preferences to be risk averse if there exists a set of probabilities which leads the individual to uniformly prefer the sure thing to non-degenerate lotteries using those probabilities. This definition of risk aversion essentially requires that indifference curves be suitably convex in the neighborhood of the certain income vector, but does not impose any strong curvature properties upon preferences. A stronger form of risk aversion is given by requiring \( e \) to be quasi-concave (Debreu, 1959; Yaari, 1969; Malinvaud, 1970). An even stronger form of risk aversion requires \( e \) to be concave. We state the following fact (Chambers and Quiggin, 2000) as a lemma for later use:

**Lemma 1** If \( e \) is quasi-concave and satisfies constant absolute risk aversion

\[ e(\mu y) \geq \mu e(y) \quad 0 < \mu < 1. \]

The combination of constant absolute risk aversion and quasi-concavity requires that the certainty equivalent be sub-homogeneous. An immediate consequence of Lemma 1 is that preferences cannot simultaneously exhibit strict quasi-concavity and constant risk aversion (Chambers and Quiggin, 2000).\(^2\) As Chambers and Quiggin (2001) observe this

\[ e(\mu y) > \mu e(y). \]

\(^2\)If preferences are strictly quasi-concave and satisfy constant absolute risk aversion, then
implies that the only class of quasi-concave preferences consistent with constant risk aversion are ones whose least-as-good sets have linear boundaries everywhere away from the sure-thing vector.

By standard duality theorems (Färe, 1988), there is a cost function dual to $X(z)$ and defined

$$c(w, z) = \min \{ wx : x \in X(z) \}$$

if $X(z)$ is nonempty and $\infty$ otherwise. The cost function satisfies:

C.1. $c(w, z)$ is positively linearly homogeneous, non-decreasing, concave, and continuous in $w$;

C.2. Shephard’s Lemma;

C.3. $c(w, z) \geq 0$, $c(w, 0_{M \times S}) = 0$, and $c(w, z) > 0$ for $z \geq 0_{M \times S}, z \neq 0_{M \times S}$;

C.4. \[\|vecz^k\| \to \infty \text{ as } k \to \infty \Rightarrow c(w, z^k) \to \infty \text{ as } k \to \infty;\]

C.5. $c(w, z)$ is convex and continuous on $\mathbb{R}^S_{++}$.

Moreover, by standard duality theorems (Färe 1988):

$$X(z) = \cap_{w > 0} \{ x : wx \geq c(w, z) \}.$$

\section{The Indirect Certainty Equivalent}

Consider the following correspondence giving feasible net return levels,

$$B(w, p, q) = \{ y : y_s \leq p_sz_s + q_s - c(w, z), \ z \in \mathbb{R}^M_{++}, s \in \Omega \}. $$

By the properties of the cost function $B$ is continuous, and $B(w, p, q)$ is a closed set. Where needed we shall strengthen these properties to include boundedness from above to ensure the existence of well-defined maxima for the producer’s problem. Moreover,

$$B(\mu w, \mu p, \mu q) = \mu B(w, p, q), \quad \mu > 0.$$ 

Our main interest is in the firm’s input and output choices. Therefore, we examine these
choices conditional on its holding of financial assets.\textsuperscript{3} The indirect certainty equivalent\textsuperscript{4} is defined

$$I(w, p, q) = \max_y \{ e(y) : y \in B(w, p, q) \}$$

$$= \max_z \{ e(pz + q - c(w, z) \mathbb{1}) \}$$ (1)

if $B(w, p, q)$ is nonempty and $-\infty$ otherwise. Both definitions of $I(w, p, q)$ prove convenient in what follows. We have (all proofs are in the appendix):

**Proposition 2** $I(w, p, q)$ is continuous in $(w, p, q)$, nondecreasing in $p$ and $q$, and non-increasing and quasi-convex in $w$.

Denote

$$z(w, p, q) \in \arg \max_z \{ e(pz + q - c(w, z) \mathbb{1}) \}.$$  

By the theorem of the maximum (Berge, p.116), the elements of $z(w, p, q)$ are upper semi-continuous. Moreover, upon applying Shephard’s lemma (Färe, 1988) to $c(w, z(w, p, q))$ in the case of an unique cost minimizing solution we obtain

$$x(w, p, q) = \nabla_w c(w, z(w, p, q))$$

where $x(w, p, q)$ is the vector of optimal input demands and $\nabla$ denotes the gradient with respect to the subscripted vector or element of the vector as appropriate. Hence, we obtain the following generalization of Hotelling’s lemma for the generalized Sandmovian model as a straightforward consequence of standard arguments in optimization theory.

**Proposition 3** If $c$ is differentiable in $w$ at $z(w, p, q)$ and $I$ is differentiable with $\nabla_q I(w, p, q) \neq 0^S$, then

$$z_s(w, p, q) = \frac{\nabla_{pz} I(w, p, q)}{\nabla_q I(w, p, q)} \quad \forall s \in \Omega$$

$$x(w, p, q) = -\frac{\nabla_w I(w, p, q)}{\nabla_q I(w, p, q) \mathbb{1}}.$$  

\textsuperscript{3}To determine the interaction between their optimal portfolio choice and their production decisions, one can always use the indirect certainty equivalent derived below to characterize optimal portfolio choice. Chambers and Quiggin (1997, 2000) examine these joint choices for the case of a single product firm facing, respectively, expected-utility and generalized Schur concave preferences.

\textsuperscript{4}Strictly speaking this is an indirect certainty equivalent conditioned on $q$. 

5
Because
\[ x(w, p, q) = \nabla_w c(w, z(w, p, q)), \]
a standard comparative-static decomposition of price effects exists for the optimal input demands. Hence, in the smooth case, the compensated input demands are downward sloping and symmetric as a consequence of the concavity of \( c(w, z) \). The overall effect of a change in an input price, as in profit maximization, can be broken into two parts, the already mentioned compensated effect and an expansion effect associated with the induced change in the state-contingent output vector. So, for example, assuming for notational simplicity that \( M = 1 \),
\[ \frac{\partial x_k(w, p, q)}{\partial w_j} = \frac{\partial^2 c(w, z)}{\partial w_k \partial w_j} + \sum_{s \in \Omega} \frac{\partial^2 c(w, z)}{\partial w_k \partial z_s} \frac{\partial z_s(w, p, q)}{\partial w_j}. \]
Unlike the case of profit maximization but similar to standard demand analysis, the overall effect need not be symmetric and input demands need not be downward sloping in their own price. These observations and straightforward manipulation can then be used to generalize the comparative-static results of Pope (1978) to the case of both price and production uncertainty.

To this point, all of our results have been obtained independent of any probability measure. However, in many instances, researchers may not be as interested in the \textit{ex post} supplies as they are in an expected supply expression.\textsuperscript{5} Presuming the existence of such a probability measure, which is known to the researcher and given by \( \pi_s, s \in \Omega \), then it is a straightforward consequence of Proposition 3 that
\[ E_x z(w, p, q) = \sum_{s \in \Omega} \pi_s \frac{\nabla_p I(w, p, q)}{\nabla q_s I(w, p, q)}. \]
In financial applications, we may be interested in the value of the firm’s output under various assumptions about securities markets. Suppose first there exists a complete set of state-contingent securities \( \rho \in \mathbb{R}^S \) and that only one commodity is produced, \( M = 1 \). Then we can write the value of the firm’s production of that output as
\textsuperscript{5}For example, econometricians studying supply and input demand response under uncertainty may not possess enough degrees of freedom to estimate the \textit{ex post} supplies accurately.
\[
\sum_{s \in \Omega} \rho_s \nabla_{p_s} I(\mathbf{w}, \mathbf{p}, \mathbf{q}) \nabla_{q_s} I(\mathbf{w}, \mathbf{p}, \mathbf{q}).
\] (3)

More generally, suppose the producer’s portfolio \( \mathbf{q} \) is made up of a set of assets \( a^1 \ldots a^K \) with the holding of asset \( k \) being denoted by \( \alpha^k \in \mathbb{R} \) (thus we allow for zero holdings and short-selling). If the return on asset \( k \) in state \( s \) is denoted by \( r^k_s \), then

\[ q_s = \sum_k \alpha^k r^k_s. \]

Let the market price of a unit of asset \( k \) (expressed in terms of the nonstochastic unit vector \( 1 \)) be \( v^k \). Then a price vector \( \mathbf{\rho} \in \mathbb{R}_{++}^S \) is a supporting state-claim price vector for the set of assets \( a^1 \ldots a^K \) if

\[ v^k = \sum_{s \in \Omega} \rho_s r^k_s \quad \forall k. \]

Now define \( \mathbf{\rho} \in \mathbb{R}^S \) such that

\[ \sum_{s \in \Omega} \rho_s = 1 \]

\[ \frac{\rho_s}{\rho_t} = \frac{c_s(\mathbf{w}, \mathbf{z})}{c_t(\mathbf{w}, \mathbf{z})} \quad \forall s, t. \]

Then, assuming the producer’s portfolio has been chosen to maximize welfare, \( \mathbf{\rho} \) is a supporting state-claim price vector satisfying (3).

### 3 Restrictions on preferences and the form of the indirect certainty equivalent

**Proposition 4** If preferences satisfy constant absolute risk aversion

\[ I(\mathbf{w}, \mathbf{p}, \mathbf{q}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}), \]

where

\[ I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}) = \max_z \{ e(pz + q) - c(\mathbf{w}, \mathbf{z}) \}, \]

and \( I^A \) is continuous in \( (\mathbf{w}, \mathbf{p}, \mathbf{q}) \), nondecreasing in \( \mathbf{p} \) and \( \mathbf{q} \), nonincreasing and convex in \( \mathbf{w} \), and

\[ I^A(\mathbf{w}, \mathbf{p}, \mathbf{q} + \delta \mathbf{1}) = I^A(\mathbf{w}, \mathbf{p}, \mathbf{q}) + \delta, \quad \delta \in \mathbb{R}. \]
Optimal input demands, as well as optimal state-contingent outputs, are therefore independent of any non-stochastic changes in wealth if preferences exhibit constant absolute risk aversion. This reiterates the classic finding from portfolio analysis that the amount of a risky asset purchased is independent of the individual’s wealth under constant absolute risk aversion. Moreover, Proposition 4 also establishes that optimal input demands are always downward sloping in their own prices and for suitably smooth cases that the sub-Hessian matrix of input demands is negative semi-definite in input prices. Furthermore,

**Corollary 5** If preferences satisfy constant absolute risk aversion and \( I^A(w, p, q) \) is differentiable in input prices

\[
e(z(w, p, q) + q) = I^A(w, p, q) - w \nabla w I^A(w, p, q).
\]

Notice the similarity between Corollary 5 and Lau’s (1978) early results on normalized profit functions. Under CARA, the producer’s problem is isomorphic to normalized profit maximization. Hence, one intuitively expects to find results on optimal input demand behavior that exactly parallel those results.

**Proposition 6** If the certainty equivalent satisfies constant relative risk aversion

\[
I(\mu w, \mu p, \mu q) = \mu I(w, p, q).
\]

An immediate consequence of Proposition 6 is that optimal state-contingent revenues and, thus, optimal input demands are homogeneous of degree zero in prices and the producer’s portfolio. Combining Propositions 4 and 6, we obtain

**Corollary 7** If preferences exhibit constant risk aversion

\[
I(w, p, q) = I^A(w, p, q)
\]

with \( I^A \) continuous and positively linearly homogeneous in \((w, p, q)\), nondecreasing in \(p\) and \(q\), nonincreasing and convex in \(w\), and

\[
I^A(w, p, q + \delta 1) = I^A(w, p, q) + \delta, \quad \delta \in \mathbb{R}.
\]
Under expected utility preferences, it is well-known that constant risk aversion is only possible if the individual is risk neutral. In that case, $I^4(w, p, q)$ would correspond to the expected profit function plus the expected value of the producer’s portfolio. However, it is also well-known that other risk-averse preference functionals can exhibit constant risk aversion. For example, maximin preferences,

$$e(y) = \min \{y_1, ..., y_s\},$$

piecewise linear preferences,

$$e(y) = \inf \{\pi y : \pi \in \mathcal{P}\},$$

where $\mathcal{P}$ is some subset of the probability simplex, and linear mean-standard deviation preferences satisfy constant risk aversion. Corollary 7 establishes that in these cases the optimal input demands for a risk averter would behave very similarly to the derived demands for a risk-neutral individual. Empirically, what appears to distinguish them the most from their risk-neutral counterparts is their dependence upon the producer’s asset portfolio.

The close connection between constant risk averse and risk-neutral preferences is particularly stark in the case of maximin preferences. For maximin preferences, optimality requires

$$p_s z_s - p_t z_t = q_t - q_s,$$

for all $s, t$. Hence, net returns are stabilized across states of Nature. Let $r$ denote the stable state-contingent revenue across states of Nature. Then the producer’s problem is one of maximizing sure net returns according to

$$V(w, q, p) = \max_{r, z} \{r - c(w, z) : p_s z_s = r - q_s, \quad s \in \Omega\}.$$

$V(w, q, p)$ can be thought of as sure-profit function. Apart from its dependence upon $q$, it is isomorphic to a normalized profit function. In addition to the results we have already established, by the convexity of the cost function in state-contingent outputs and standard arguments in optimization theory, $V(w, q, p)$ will be concave in $q$.\footnote{Because the preferences are defined as the pointwise minimum of a set of concave functions, the certainty equivalent is concave (Rockafellar, 1970). Hence, concavity here is also a direct consequence of the concavity results reported below.} Hence, changes in
the distribution of the producer’s asset portfolio affect producer choices, but because preferences are consistent with constant absolute risk aversion, nonstochastic changes in the portfolio have no effect on these decisions. Empirically, this observation suggests that one will not be able to discriminate between risk neutral and completely risk-averse preferences unless one possesses information on the producer’s portfolio in addition to information on his predetermined wealth. This implication is of considerable interest because since the time of Sandmo (1971), it has become routine to identify, for example, input responsiveness to wealth changes as an identifying characteristic of risk-averse behavior.

We now turn attention to quasi-concave certainty equivalents.

**Proposition 8** If 𝑒 is quasi-concave, \(I(w, p, q)\) is quasi-concave in \(q\).

Proposition 8 establishes that the indirect certainty equivalent inherits the producer’s basic risk aversion. Because the set

\[\{q : I(w, p, q) \geq i\}\]

is convex, there now always exist a supporting hyperplane, \(\mathbf{h} \in \mathbb{R}^q_+\), to

\[\{q : I(w, p, q) \geq I(w, p, \mu 1)\}\]

such that

\[\hat{q} \in \{q : I(w, p, q) \geq I(w, p, \mu 1)\} \implies \sum_s h_s \hat{q}_s \geq \mu \sum_s h_s.\]

Thus,

\[\hat{a}_s = \frac{h_s}{\sum_k h_k}, \quad s \in \Omega\]

can be interpreted as a set of risk-neutral probabilities for which the producer exhibits aversion to risk in the financial assets in the sense of Quiggin and Chambers (1998). For these probabilities, the producer always weakly prefers the non-stochastic portfolio, \(\mu 1\), to any portfolio with the same expected value. When \(I(w, p, q)\) is smoothly differentiable, these probabilities are unique and given by

\[\hat{a}_s = \frac{\nabla_q I(w, p, \mu 1)}{\nabla_q I(w, p, \mu 1) 1}, \quad s \in \Omega.\]

\(^7\)There can be more than one.
The producer’s risk-neutral (virtual) probabilities in the smooth case are obtained similarly as

$$\pi^n_s = \frac{\nabla_q I(w, p, q)}{\nabla_q I(w, p, q) 1}, \quad s \in \Omega.$$ 

**Proposition 9** If $e$ is concave, $I$ is concave in $q$.

By Proposition 4 and Lemma 1

**Corollary 10** If $e$ is quasi-concave and satisfies constant absolute risk aversion

$$I(w, p, q) = I^A(w, p, q),$$

where

$$I^A(w, p, q) = \max \{ e(pz + q) - c(w, z) \},$$

where $I^A$ is continuous in $(w, p, q)$, nondecreasing in $p$, nondecreasing and quasi-concave in $q$, nonincreasing and convex in $w$, with

$$I^A(w, p, q + \delta 1) = I^A(w, p, q) + \delta, \quad \delta \in \mathbb{R},$$

and

$$I^A(w, p, \mu q) \geq \mu I^A(w, p, q) \quad 0 < \mu < 1.$$

4 Concluding Remarks

We have developed the general properties of the indirect certainty equivalent and demonstrated a generalized version of Hotelling’s lemma for producers facing both price and production uncertainty independent of any assumptions about the producer’s attitudes toward risk and independent of any notion of probabilities. We have also examined the structural consequences for the indirect certainty equivalent of various restrictions on producer preferences.

Global duality theorems seem difficult to obtain in our problem setting and thus none have been presented. However, local duality correspondences can be obtained straightforwardly by applying the general theorems of Epstein (1981) and Diewert (1982) to this specific case after imposing some additional structure on $B(w, p, q)$. 
5 References


6 Appendix: Proofs

Proof Proposition 2: Continuity follows by the theorem of the maximum (Berge, 1963, p.116) and the properties of $B$. The monotonicity properties are trivial. Let
\[
\tilde{z} = z \left( \lambda w^o + (1 - \lambda) w^1, p, q \right).
\]

Quasi-convexity in $w$ is established by
\[
I \left( w, p, \lambda w^o + (1 - \lambda) w^1 \right) = e \left( p \tilde{z} + q - c \left( \lambda w^o + (1 - \lambda) w^1, \tilde{z} \right) \right)
\leq e \left( p \tilde{z} + q - \lambda c(w^o, \tilde{z}) - (1 - \lambda) c(w^1, \tilde{z}) \right)
\leq e \left( p \tilde{z} + q - \min \left\{ c(w^o, \tilde{z}), c(w^1, \tilde{z}) \right\} \right),
\]
for $0 < \lambda < 1$. The first inequality follows by C.1 (concavity in $w$).

Proof of Proposition 4: By constant absolute risk aversion
\[
e (pz + q - c) = e (pz + q) - c.
\]

Continuity follows from the theorem of the maximum, and the monotonicity properties are obvious. By the concavity of $c(w, z)$ in $w$ (C.1), $e (pz + q) - c(w, z)$ is convex in $w$. Hence, $I(w, p, q)$ is the pointwise supremum of a series of convex functions and thus convex in $w$ by Theorem 5.5 of Rockafellar (1970).

Proof of Proposition 6:
\[
I(\mu w, \mu p, \mu q) = \max_z \{ e (\mu p z + \mu q - c(\mu w, z)) \}
\]
\[
= \max_z \{ e (\mu p z + \mu q - \mu c(w, z)) \}
\]
\[
= \mu \max_z \{ e (pz + q - c(w, z)) \},
\]
under constant relative risk aversion.

Proof of Proposition 8: Let
\[
z^0 = z \left( w, p, q^0 \right), z^1 = z \left( w, p, q^1 \right)
\]
\[ \hat{z} = \lambda z^0 + (1 - \lambda) z^1, \]
\[ y^0 = pz^0 + q^0 - c(w, z^0)1, \]
\[ y^1 = pz^1 + q^1 - c(w, z^1)1. \]

Then for \(0 < \lambda < 1,\)
\[
I\left( w, p, \lambda q^0 + (1 - \lambda) q^1 \right) \geq e\left( p\hat{z} + \lambda q^0 + (1 - \lambda) q^1 - c(w, \hat{z})1 \right) \\
\geq e\left( \lambda y^0 + (1 - \lambda) y^1 \right) \\
\geq \min\{e\left( y^0 \right), e\left( y^1 \right)\}.
\]

The second inequality follows by C.5 (convexity), and the third by quasi-concavity of \(e.\)

**Proof** of Proposition 9: Replace last line of the proof of Proposition 8 by
\[ \lambda e\left( y^0 \right) + (1 - \lambda) e\left( y^1 \right) \]

which follows by concavity of \(e.\)