Increasing Uncertainty: A Definition

Abstract

We present a definition of increasing uncertainty, in which an elementary increase in the uncertainty of any act corresponds to the addition of an 'elementary bet' that increases consumption by a fixed amount in (relatively) 'good' states and decreases consumption by a fixed (and possibly different) amount in (relatively) 'bad' states. This definition naturally gives rise to a dual definition of comparative aversion to uncertainty. We characterize this definition for a popular class of generalized models of choice under uncertainty and briefly outline the implications of our work for comparative static analysis of problems in finance theory and environmental economics.

Keywords: uncertainty, ambiguity, risk, non-expected utility

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1 Introduction

Most formal analysis of economic decisions under uncertainty has relied on concepts of subjective probability. Significant advances in the discussion of preferences in the absence of well-defined subjective probabilities, and in understanding the relationship between preferences and subjective probabilities, have been made by Schmeidler (1989), Machina and Schmeidler (1992), Epstein (1999) and Ghirardato and Marinacci (2002). A particularly important contribution is that of Epstein and Zhang (2001), who set out a range of desiderata for a definition of ambiguity, and provide a definition meeting most of these desiderata.

The analysis of economic decisions in the absence of well-defined subjective probabilities has often been referred to in terms of Knight’s (1921) distinction between risk and uncertainty. However, Knight’s discussion of the role of insurance companies and the Law of Large Numbers makes it clear that his conception of risk was confined to cases where objective probabilities can be defined in frequentist terms, and where risk can effectively be eliminated through pooling and spreading. All other cases, including those where individuals possess personal subjective probabilities, were effectively classed by Knight as involving uncertainty. The distinction now commonly drawn between ‘risk’ and ‘uncertainty’ could not be developed properly until the formulation of well-defined notions of subjective probability by de Finetti (1937) and Savage (1954).

The first writer to clearly identify cases where preferences were inconsistent with first-order stochastic dominance, relative to any possible probability distribution, was Ellsberg (1961) who distinguished between risk (subjective probabilities satisfying the Savage axioms) and ambiguity, leaving uncertainty as a comprehensive term. Therefore, consistent with the usage of Savage and Ellsberg, and with usage in the general economics literature, we will use the term uncertainty to encompass all decisions involving non-trivial state-contingent outcome vectors, whether or not the preferences and beliefs associated with these decisions can be characterized by well-defined subjective probabilities. Events for which subjective probabilities are (respectively, are not) well-defined will be referred to as ‘unambiguous’ (respectively, ‘ambiguous’) and problems involving acts measurable with respect to unambiguous events will be said to involve ‘risk’. Our usage is consistent with Ghirardato and Marinacci (2002) and Epstein and Zhang (2001).

Epstein and Zhang (2001) provide a rigorous definition of ambiguous and unambiguous events, and lay the basis for an analysis of preferences under uncertainty, including both risk
and ambiguity. In light of this, the definition proposed by Epstein (1999) for a comparative ambiguity aversion relation over preference relations can now be stated in a solely preference-based and functional-form free manner. However, questions of when one act is more uncertain or more ambiguous than another are not addressed in these analyses, except in the polar case where one act is ambiguous and the other is unambiguous. Ghirardato and Marinacci (2002) propose a model-free definition of comparative uncertainty aversion: one preference relation is more uncertainty averse than another, if whenever the latter relation expresses a weak preference for a constant act (that is, one that will yield the same outcome no matter what state of the world will obtain) over another act, then so must the former relation. They do not, however, consider the question of when one act is more uncertain than another except in the polar case where one of the acts yields a certain outcome.

By contrast, the concept of an increase in risk, and the economic consequences of increases in risk, have been analyzed extensively, beginning with the work of Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970). These authors independently derived and characterized the second-order stochastic dominance condition (in terms of mean-preserving spreads), under which all risk-averse expected utility maximizers will prefer one probability distribution to another. Quiggin (1993) introduced an alternative notion of monotone (mean-preserving) increase in risk, defined in terms of co-monotonic random variables instead of mean-preserving spreads. Landsberger and Meilijson (1994) pointed out that this notion of increase in risk coincides with the Bickel and Lehmann (1976) notion of dispersion of random variables with equal means. Yaari (1969) argued that since any lottery is by definition a ‘mean-preserving spread’ of its mean, the weakest notion of risk aversion simply requires that the mean of a lottery for sure is weakly preferred to the lottery itself. Subsequent studies examined a wide range of generalizations of these stochastic dominance conditions, typically associated with more restrictive conditions on utility functions. Other papers that have extensively analyzed the concept of increasing risk in the context of generalized expected utility models include Chew, Karni and Safra (1987), Chateauneuf, Cohen and Meilijson (1997), Grant, Kajii and Polak (1992), Quiggin (1993) and Safra and Zilcha (1989).

1 To the best of our knowledge, the only other definition based solely on preferences is that provided by Nehring (2001). In other papers such as Gilboa and Schmeidler (1994), Mukerji (1997), Sarin and Wakker (1998), Ghiradato and Marinacci (2002), Nehring (1999) and Ryan (2001), the analysis focuses on a class of preference relations that admit a specific functional form. The criteria for what constitutes an ambiguous or unambiguous event is then defined in terms of a property or properties of the specific functional form representation that each of these preference relations admit.
Most concepts of increasing risk that have been considered in the literature are inherently dependent on the existence of well-defined subjective probabilities. This is obviously true of mean-preserving increases in risk, since the mean depends on probabilities. Even notions such as that of a compensated increase in risk (Diamond and Stiglitz, 1974), which do not depend on mean values, incorporate probabilities in their definitions. Yet the intuitive concept of an increase in the uncertainty of a prospect does not seem to depend crucially on probabilities. To take a simple example, doubling the stakes of a bet surely increases the uncertainty associated with that bet, regardless of whether the parties have well-defined and common subjective probabilities regarding the event that is the subject of the bet.

The main object of this paper is to examine concepts of increasing uncertainty, that are independent of any notion of subjective probabilities. A natural starting point is to consider whether existing concepts of ‘elementary mean-preserving increases in risk’, such as monotone spreads and Dalton transfers yield useful results when reference to probability distributions and means is dropped. We show that the monotone spread concept is robust to this generalization, but that concepts based on Dalton transfers, including the Rothschild-Stiglitz definition of increasing risk, have no content in the absence of well-defined probabilities.

Any definition of increasing uncertainty naturally gives rise to a dual definition of comparative aversion to uncertainty. We characterize this definition for a popular class of generalized models of choice under uncertainty.

An important objective of this work is to extend the economic applicability of the concepts developed by previous writers on this topic. Despite significant progress in characterizing preferences under uncertainty, without reliance on probability concepts, there has been relatively little analysis of the economic choices under uncertainty. One important problem is that comparative static analysis requires the adoption of some notion of an increase in uncertainty, and there is no generally agreed concept of an increase in uncertainty. We briefly outline the implications of our work for comparative static analysis of problems in finance theory and environmental economics. Proofs of the results, unless otherwise stated, appear in the appendix.

2 Preliminaries

Set-up and Notation. Denote by \( S = \{ \ldots, s, \ldots \} \) a set of states and \( \mathcal{E} = \{ \ldots, A, B, \ldots, E, \ldots \} \) the set of events which is a given \( \sigma \)-field on \( S \). We take the set of outcomes to be the set of non-negative real numbers, or ‘consumption levels’. An act is a (measurable) real-valued
and bounded function \( f : \mathcal{S} \rightarrow \mathbb{R}_+ \). Let \( f(\mathcal{S}) = \{ f(s) \mid s \in \mathcal{S} \} \) be the outcome set associated with the act \( f \), that is, the range of \( f \). Let \( \mathcal{F} = \{ \ldots, f, g, h, \ldots \} \) denote the set of acts on \( \mathcal{S} \); and let \( \mathcal{F}_0 \) denote the set of simple acts on \( \mathcal{S} \); that is, those with finite outcome sets. We will abuse notation and use \( x \) to denote both the outcome \( x \) in \( \mathbb{R}_+ \) and the constant act with \( f(\mathcal{S}) = \{ x \} \).

The following notation to describe an act will be convenient. For an event \( E \) in \( \mathcal{E} \), and any two acts \( f \) and \( g \) in \( \mathcal{F} \), let \( f \llsg \) be the act which gives, for each state \( s \), the outcome \( f(s) \) if \( s \) is in \( E \) and the outcome \( g(s) \) if \( s \) is in the complement of \( E \) (denoted \( \mathcal{S} \setminus E \)).

In general, for any finite partition \( \mathcal{P} := \{ A^1, \ldots, A^n \} \) of \( \mathcal{S} \) and any list of \( n \) acts \( (h^1, \ldots, h^n) \), let \( h_1^1, h_2^2, \ldots, h_{A^n-1}^{n-1}, h^n \) be the act that yields \( h^i_j \) if \( s \) is in \( A^j \).

Let \( \succsim \) be a binary relation over \( \mathcal{F} \), representing the individual’s preferences. Let \( \succ \) and \( \sim \) correspond to strict preference and indifference, respectively.

Given \( \succsim \), for any act \( f \) in \( \mathcal{F} \), we define the ‘at least as good as \( f \)’ set as the set \( \succsim_f = \{ g \in \mathcal{F} : g \succsim f \} \).

An event \( E \) is deemed null for the preference relation \( \succsim \), if for all \( f \) and \( g \) in \( \mathcal{F} \), \( f \llsg \sim g \).

We say a sequence of acts \( f_n \) converges point-wise in the limit to \( f \), written \( f_n \rightharpoonup f \), if, for each \( s \) in \( \mathcal{S} \), the sequence of real numbers, \( f_n(s) \) converges to \( f(s) \).

The only maintained assumptions we make on this preference relation is that it is a (point-wise) continuous preference ordering and satisfies a weak form of monotonicity.

**Axiom 1** The preference relation \( \succsim \) is a continuous weak order: that is, it is transitive and complete and, for any of sequences of acts \( \langle f_n \rangle \) and \( \langle g_n \rangle \), such that \( f_n \rightharpoonup f \) and \( g_n \rightharpoonup g \), if \( f_n \succsim g_n \) for all \( n \), then \( f \succsim g \).

**Axiom 2** The preference relation \( \succsim \) is monotonic. That is, if for any pair of acts, \( f \) and \( g \) in \( \mathcal{F} \), \( f(s) \geq g(s) + \varepsilon \), with \( \varepsilon > 0 \), for all \( s \) in \( \Omega \), then \( f \succ g \).

We observe that any preference relation \( \succsim \) on \( \mathcal{F} \) satisfying the axioms above may be characterized by a unique certainty equivalent of the form

\[
m(f) = \sup\{ x \in \mathbb{R}_+ : f \succsim x \}.
\]

## 2.1 An elementary increase in uncertainty

Under what circumstances may we view one act as being more uncertain than another? Given a probability measure exogenously defined over the state space, it seems uncontroversial to denote any act as more risky than the constant act which yields the mean outcome
of that act (evaluated according to that probability distribution) in every state. Other statistical partial orderings, such as second-order stochastic dominance or the Rothschild-Stiglitz definition of more risky, can also be invoked. However, in the absence of exogenously given probabilities, it seems more natural to build up a ‘more-uncertain-than’ partial ordering over acts by considering the simplest operation that can be performed on an act that unequivocally increases the uncertainty associated with that act. The most elementary operation that we believe unequivocally increases the uncertainty associated with an act, is one that involves adding an ‘elementary bet’ to that act. The addition of an elementary bet increases consumption by a fixed amount in the (relatively) ‘good’ states and decreases consumption by a fixed (and possibly different) amount in the (relatively) ‘bad’ states. We refer to the addition of such a cocomonotonic elementary bet as an elementary increase in uncertainty.

**Definition 1** Fix a pair of acts $f, g \in \mathcal{F}$. The act $g$ represents an elementary increase in uncertainty of the act $f$, denoted $g \uparrow f$, if there exists a pair of positive numbers $\alpha$ and $\beta$, and an event $E^+ \in \mathcal{E} \setminus \{S, \emptyset\}$ such that: (i) for all $s$ in $E^+$, $g(s) - f(s) = \alpha$; (ii) for all $s$ in $S \setminus E^+$, $f(s) - g(s) = \beta$; and (iii) $\sup \{f(s) : s \in S \setminus E^+\} \leq \inf \{f(s) : s \in E^+\}$.

Correspondingly, we define a notion of comparative uncertainty aversion:

**Definition 2** Fix $\succcurlyeq$ and $\succcurlyeq\$. The preference relation $\succcurlyeq$ is at least as uncertainty averse at $f$ as $\succcurlyeq\$ if for any $g \uparrow f$, $f \succcurlyeq\$ $g$ implies $f \succcurlyeq g$. The preference relation $\succcurlyeq$ is everywhere at least as uncertainty averse as $\succcurlyeq\$ if for all $f$, $\succcurlyeq$ is as least as uncertainty averse at $f$ as $\succcurlyeq\$.

Before we explore in more detail the implications of this definition, let us evaluate it against the desiderata set out in Epstein and Zhang (2001). Epstein and Zhang argue that a definition should be:

D1. Behavioral or expressed in terms of preferences: that is, verifiable in principle given suitable data on behavior.

D2. Model-free: since concepts of uncertainty and ambiguity are more basic than specifications of preferences, a definition should not be tied to any particular model.

D3. Explicit and constructive: Given an event, it should be possible to check whether or not it is ambiguous.

D4. Consistent with probabilistic sophistication on unambiguous events.

Since, wherever relevant, we employ the Epstein–Zhang characterization of ambiguous and unambiguous events, our definition of increasing uncertainty automatically inherits property D4. Also, as with Epstein and Zhang, the degree to which D1 is satisfied is limited
by the assumption of an objectively given state-space over which acts are defined as mappings from that state-space to an outcome space. Our main concern, therefore relates to the properties D2 and D3, and particularly with D2, the requirement that the characterization of increasing uncertainty should be model-free. Subject only to the assumption of an objectively given state-space and state-independent preferences, our definition of an elementary increase in uncertainty is model-free. This model-free status carries over to the definition of comparative uncertainty aversion.

3 Increases in uncertainty and uncertainty aversion

Our first observation about the definition of an elementary increase in uncertainty is that, no matter what assessment an individual attaches to any event (that may incorporate his or her belief and/or decision weight), an elementary increase in uncertainty always reduces consumption in the worst event and increases consumption in the best event. Furthermore, if $gUf$ then $g$, $f$ and the function $g - f$ are pairwise co-monotonic functions. That is, for every pair of states $s, t \in S$,

$$(g(s) - g(t))(f(s) - f(t)) \geq 0$$

$$(g(s) - f(s) - g(t) + f(t))(f(s) - f(t)) \geq 0$$

$$(g(s) - g(t))(g(s) - f(s) - g(t) + f(t)) \geq 0.$$  

As nothing in the inequalities in the above inequalities require the acts in question to be simple, we shall adopt these inequalities to define the more uncertain relation between any pair of acts.

**Definition 3** Fix a pair of acts $f, g \in \mathcal{F}$. The act $g$ is more uncertain than the act $f$, denoted $gUf$, if there exists a real-valued function $h$ on $S$, comonotonic with $f$ such that $\sup h > 0$, $\inf h < 0$ and $g = f + h$.

Our main result in this section is that the relation $U$ is simply the transitive continuous closure of the relation $U$.

**Proposition 3** Fix a pair of acts $f, g \in \mathcal{F}$. If $gUf$ then there exist sequences of simple acts, $\langle f_n \rangle$ and $\langle g_n \rangle$, such that $f_n \to f$ and $g_n \to g$, and for each $n$ there exists a finite sequence of simple acts $\langle h^n_{m/n} \rangle_{m=1}^{M^n}$, such that $h^n_1 = f_n$, $h^n_{M^n} = g_n$ and $h^n_{m+1}Uh^n_m$, $m = 1, \ldots M^n - 1$. 

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The following is an immediate corollary of Proposition (3).

**Corollary 4** Fix $\succsim$ and $\approx_r$. The preference relation $\succeq$ is everywhere at least as uncertainty averse as $\approx_r$, if and only if,

$$f \approx_r g \text{ implies } f \succeq g \text{ for all } g \forall f.$$ 

Also, we obtain

**Corollary 5** Any act $f$ is more uncertain than its certainty equivalent $m(f)$.

**Corollary 6** If $\succsim$ is everywhere at least as uncertainty averse as $\approx_r$, then for any $f$

$$m(f) \leq \hat{m}(f).$$

From Corollary 6 it follows that if $\succsim$ is everywhere at least as uncertainty averse as $\approx_r$ then $\succeq$ is more uncertainty averse than $\approx_r$ in the weaker sense of the following definition proposed by Ghirardato and Marinacci’s (2002): the preference relation $\succeq$ is more (weakly) uncertainty averse than $\approx_r$ if for any act $f$ and any constant act $x$,

$$x \succeq f \Rightarrow x \approx_r f \text{ and } x \succeq f \Rightarrow x \succeq f$$

Ghirardato and Marinacci argue that their definition only relies upon the weakest pre-judgement about what constitutes an unambiguous act, namely one that yields a given outcome for certain. Our definition encompasses this but goes further. Our rationale is that the adding to an act a comonotonic simple bet should be considered by construction to have increased its uncertainty. Hence the natural definition for comparative uncertainty is the stronger one we propose in which a comonotonic simple bet being viewed unfavorably by an individual should entail that it is viewed unfavorably by any other individual who is more uncertainty averse.

Epstein (1999) proposed a definition of comparative ambiguity aversion that explicitly controled for ‘risk aversion’. He did this by assuming that there was a rich set of exogeneously defined ‘unambiguous events’ $\mathcal{A} \subset \mathcal{E}$, that was closed under complementation and union. Any act that was measurable with respect to $\mathcal{A}$ was deemed an unambiguous act. The preference relation $\succeq$ is more ambiguity averse than $\approx_r$ if for every unambiguous act $h$ and every act $f$ 

$$h \succeq f \Rightarrow h \approx_r f \text{ and } h \succeq f \Rightarrow h \succeq f$$

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Adopting Epstein and Zhang’s (2001) purely behavioral definition for an unambiguous event, allows the outside analyst to compare two preference relations according to Epstein’s definition without having to assume a priori which events are ambiguous or unambiguous.\footnote{Epstein and Zhang (2001) define an event $T$ to be unambiguous if (a) for all disjoint subevents $A, B \subset S \setminus T$, acts $h$, and outcomes $x, x', z, z'$, $x_A x_B z_T h \succeq x_A x_B z_T h$ implies $x_A x_B z_T h \succeq x_A x_B z_T h$ and (b) the condition obtained if $T$ is everywhere replaced by $S \setminus T$ in (a) is also satisfied. Otherwise, $T$ is ambiguous.} We do not deny the usefulness of such an isolation of ambiguity aversion from risk aversion where it can be achieved. But in circumstances where such a separation is not feasible, we believe there are useful insights and economic implications that can be drawn when comparisons according to the ‘total’ uncertainty aversion are made according to our definition of relative aversion to the addition of simple comonotonic bets.

### 3.1 Special cases

The definitions of comparative uncertainty, and of comparative uncertainty-aversion, presented above, do not depend on any specific features of the form of representation that a family of preference relations may or may not admit. It is of interest, however, to consider the case when preferences may be represented by some specific model, to characterize the relationship $\succeq$ is everywhere at least as uncertainty averse as $\hat{\succeq}$ in terms of the parameters of that model, and, where appropriate, to compare that characterization to existing results on comparative risk aversion. We begin by demonstrating that the usual characterization of comparative risk aversion for subjective expected utility is consistent with our definition. This reflects the fact that our approach satisfied the Epstein and Zhang desideratum D4. More substantively, we analyze the cases of disappointment aversion (Gul 1991), and of Choquet Expected Utility preferences (Schmeidler 1989), incorporating such important special cases as rank-dependent expected utility under risk (Quiggin 1993) and the dual model of Yaari (1987).

#### 3.1.1 Subjective Expected Utility

Let us consider the case when $\succeq$ and $\hat{\succeq}$ satisfy the assumptions of Savage’s theory of subjective expected utility (SEU). That is, assume both preference relations can be represented by certainty equivalent functionals $m, \hat{m}$ of the form

$$m(f) = u^{-1}\left(\int_s u(f(s))\pi(ws)\right) \quad \text{and} \quad \hat{m}(f) = \hat{u}^{-1}\left(\int_s \hat{u}(f(s))\hat{\pi}(ds)\right),$$

$$8$$
where $\pi$ and $\hat{\pi}$ are countably-additive and convex-ranged probability measures defined over $\mathcal{E}$, and $u$ and $\hat{u}$ are von Neumann-Morgenstern utility functions defined over $\mathcal{X}$.

The same set of necessary and sufficient conditions that are required for one preference relation to be at least as risk averse (in the sense of Rothschild and Stiglitz, 1970) as another are also necessary and sufficient for one to be at least as uncertainty averse as another.

**Proposition 7** Suppose $\succeq$ and $\tilde{\succeq}$ both admit SEU certainty equivalent representations $m(\cdot)$ and $\tilde{m}(\cdot)$, with associated probability measure and utility function pairs, $(\pi, u)$ and $(\tilde{\pi}, \tilde{u})$, respectively. Then, $\succeq$ is everywhere at least as uncertainty averse as $\tilde{\succeq}$ if and only if $\pi(A) = \hat{\pi}(A)$ for all $A \in \mathcal{E}$, and $u$ is a concave transform of $\tilde{u}$.

If we identify an SEU-maximizer with a linear utility index as being risk neutral (with respect to $\pi$), then an immediate corollary of Proposition (7) is that a necessary and sufficient condition for an SEU-maximizer to be averse to monotone mean-preserving spreads (with respect to $\pi$) is that his utility function is concave. And without requiring any other restrictions, we also know that his preference relation would agree with the partial ordering of second-order stochastic dominance (or equivalently, he is averse to all mean-preserving spreads).

These results are not surprising since it is well-known that under the expected utility model for decision making under risk (with exogenously specified probabilities) a decision maker is risk-averse in the weakest sense of always (weakly) preferring the mean of a lottery for sure to the lottery itself if and only if his utility index is concave. Such a coincidence of conditions necessary and sufficient for these three distinct notions of risk aversion (and their uncertainty analogs) does not hold in general for non-EU models of decision making under risk and non-SEU models of decision making under uncertainty. This point is illustrated by the following examples.

### 3.1.2 Disappointment Aversion

Disappointment aversion (Gul 1991) is the most widely used non-EU model displaying the “betweenness property” (see Chew 1983, Dekel 1986). In the context of the Savage framework a subjective disappointment aversion (SDA) functional representation, $V(f)$ may be implicitly defined by the equation

$$\sum_{x \in \mathbb{R}_+} \varphi\left(x, f^{-1}(x) \mid V(f)\right) = 0.$$

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3 Formal definitions in the Savage-act framework of monotone mean-preserving spreads and the partial ordering of second-order stochastic dominance are provided in the next subsection.
where

\[ \varphi(x, E, v) = \begin{cases} 
\mu(E)(1 - b)(u(x) - v) & \text{if } u(x) \geq v \\
\mu(E)(u(x) - v) & \text{if } u(x) < v
\end{cases}, \]

where \( \mu(.) \) is a probability measure defined on \( \mathcal{E} \),

\( u(.) \) is an increasing (utility) index and \( b < 1 \).

Notice that SEU is the special case in which \( b = 0 \). Gul implicitly assumes common subjective beliefs (i.e. \( \pi = \tilde{\pi} \)).

Gul (1991, Theorem 5, p676) shows that if \( b \geq \tilde{b} \) and \( u \) is a concave transformation of \( \tilde{u} \) then \( \succcurlyeq \) is at least as risk averse (in the Yaari sense) as \( \tilde{\succcurlyeq} \). It is straightforward to find counter-examples demonstrating that the converse does not hold.

As was the case for SEU, the same set of necessary and sufficient conditions that are required for one SDA preference relation to be at least as risk averse (in the sense of Rothschild and Stiglitz, 1970) as another are also necessary and sufficient for one SDA preference relation to be at least as uncertainty averse as another. To obtain a characterization of the necessary and sufficient conditions for comparative risk aversion in the sense of Rothschild and Stiglitz it is useful to define the following property

**Definition 4 (Unboundedness)** For any act \( f \), outcome \( c \) and non-null event \( E \), there exists an outcome \( d \) sufficiently large that

\[ m(f_E d) \geq c \]

For the class of unbounded SDA preferences we can show the following equivalences hold.

**Proposition 8** Suppose \( \succcurlyeq \) and \( \tilde{\succcurlyeq} \) both satisfy Unboundedness and admit subjective disappointment-aversion representations characterized by the two probability measure, utility function and disappointment parameter triples, \( (\pi, u, b) \) and \( (\tilde{\pi}, \tilde{u}, \tilde{b}) \), respectively. Then, assuming \( \succcurlyeq \) and \( \tilde{\succcurlyeq} \) are distinct, the following three statements are equivalent:

1. \( \succcurlyeq \) is everywhere at least as uncertainty averse as \( \tilde{\succcurlyeq} \);

2. \( \succcurlyeq \) is at least as risk averse (in the Rothschild-Stiglitz sense) as \( \tilde{\succcurlyeq} \);
3. \( \pi(A) = \tilde{\pi}(A) \) for all \( A \in \mathcal{E} \), \( u \) is a concave transform of \( \tilde{u} \), \( b > \hat{b} = 0 \)

Notice in particular, that comparisons of uncertainty aversion and hence comparisons of risk aversion in the Rothschild-Stiglitz sense are only possible when one of the preference relations is SEU. This result does not depend on the existence of ambiguous events, and may therefore be seen as a limitation of SDA as a model of choice under risk.

### 3.1.3 Choquet Expected Utility

The other main direction for generalizing subjective expected utility has been the so-called “rank-dependent theories” of which Choquet Expected Utility (CEU) is the most widely applied. Associated with a CEU representation is an increasing utility index \( u : X \to \mathbb{R} \) and a capacity, \( \nu \) where a capacity is a function \( \nu : \mathcal{E} \to [0, 1] \) satisfying (i) for all \( A, B \in \mathcal{E} \):

\( \nu(A) \leq \nu(B) \), (ii) for any \( \nu(\emptyset) = 0 \); and (iii) \( \nu(\mathcal{S}) = 1 \). For such a CEU-maximizer such that \( f \succeq g \) if and only if

\[
\int_{-\infty}^{\infty} [\nu(\{s : u(f(s)) \geq w\}) - \nu(\{s : u(g(s)) \geq w\})]dw \geq 0
\]

Applying (1), we can obtain the following sufficient condition.\(^4\)

**Proposition 9** For one CEU maximizer, \((u, \nu)\), with concave and differentiable \( u \), to be at least as uncertainty averse as another CEU maximizer, \((\tilde{u}, \tilde{\nu})\), with concave and differentiable \( \tilde{u} \), it is sufficient that for any event \( E \in \mathcal{E} \) and any four outcomes \( x^1 \geq x^2 \geq x^3 \geq x^4 \)

\[
\frac{\tilde{u}'(x^1)\tilde{\nu}(E)}{u'(x^4)(1 - \tilde{\nu}(E))} \geq \frac{u'(x^2)\nu(E)}{u'(x^3)(1 - \nu(E))}
\]

or equivalently,

\[
\lim_{\tilde{y} \to \infty} \left( \frac{\tilde{u}'(\tilde{y})}{u'(0)} \right) \geq \sup_{(E \in \mathcal{E})} \left( \frac{\nu(E) / (1 - \nu(E))}{\tilde{\nu}(E) / (1 - \tilde{\nu}(E))} \right).
\]

This is similar to the condition that Chateauneuf, Cohen and Meilijson (1997) derive in the context of decision making under risk for a RDEU expected utility maximizer to be averse to every monotone increase in risk. Adapting their terminology, the left-hand side

\(^4\) A model that is closely related to CEU is cumulative prospect theory (CPT). It is more general as it allows for reference-dependence. Utility is defined on deviations from a ‘status-quo’ outcome and the capacity exhibits ‘sign-dependence’, depending on whether the best outcome on an event is better or worse than the status quo. But modulo the necessary adjustments for reference-dependence, analogous results to the ones we derive for the CEU model hold for CPT.
expression in (3) may be viewed as a measure of the *greediness* of the utility function, \( \hat{u} \). The ratio \( \nu(E) / (1 - \nu(E)) \) may be interpreted as a measure of the *optimism* of the capacity \( \nu \) about the event \( E \) obtaining. Hence the right-hand side expression measures the relative optimism of the capacity \( \nu \) over the capacity \( \hat{\nu} \) about the event \( E \). Thus (3) states that a sufficient condition for \( (u, \nu) \) to be at least as uncertainty averse as \( (\hat{u}, \hat{\nu}) \) is that the latter’s greediness is never less than the former’s relative optimism over any event.

A straightforward corollary of Proposition 9 is the following.

**Corollary 10** For one CEU maximizer, \( (u, \nu) \), with concave and differentiable \( u \), to be at least as uncertainty averse as another CEU maximizer, \( (\hat{u}, \hat{\nu}) \), with concave and differentiable \( \hat{u} \), it is sufficient that \( u \) is a concave transformation of \( \hat{u} \) and for all \( E \), \( \nu(E) \leq \hat{\nu}(E) \).

Now consider expanding the class of CEU maximizers in such a way that neither \( u \) nor \( \hat{u} \) need be concave. By similar reasoning as was used to derive (2) above, we obtain the following sufficient condition:

\[
\inf \left\{ \begin{array}{l}
\hat{y} \geq \hat{x}, \ y \geq x \\
\min (\hat{y}, y) \geq \max (\hat{x}, x)
\end{array} \right\} \left( \frac{\hat{u}'(\hat{y})}{\hat{u}'(\hat{x})} \right) \geq \sup_{(E \in \mathcal{E})} \left( \frac{\nu(E) / (1 - \nu(E))}{\hat{\nu}(E) / (1 - \hat{\nu}(E))} \right). \tag{4}
\]

Furthermore, if either \( u \) or \( \hat{u} \) is linear (that is, one of the individuals is a “Yaari-CEU maximizer”) then it readily follows from (1) that (4) is both necessary and sufficient. One implication of this result is that a CEU maximizer with a non-concave utility index can be more uncertainty averse than a Yaari-CEU maximizer (or even CEU maximizer with a strictly concave utility index) provided the degree of ‘pessimism’ embodied in his capacity, as measured by the ratio \( (1 - \nu(E)) / \nu(E) \), is sufficiently strong enough to outweigh any region of non-diminishing marginal utility. Again, this accords with similar results derived in the context of decision making under risk for RDEU maximizers.

A particularly interesting application of (3), is in the context of Epstein and Zhang’s (2001) model of a CEU maximizer, \( (u, \nu) \), for whom, just from the behavioral implications of the preference relation, an outside analyst is able to classify each event as being either ‘ambiguous’ or ‘unambiguous’ for that preference relation.\(^5\) Let \( \mathcal{E}^{U/A} \subset \mathcal{E} \), denote the set

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\(^5\) Recall Epstein and Zhang (2001) define an event \( T \) to be *unambiguous* if (a) for all disjoint subevents \( A, B \subset \mathcal{S}(T) \), acts \( h, \) and outcomes \( x^*, x, z, z' \), \( x_A x_B z_T h \succeq x_A x_B z''_T h \) implies \( x_A x_B z''_h \succeq x_A x_B z''_h \) and (b) the condition obtained if \( T \) is everywhere replaced by \( \mathcal{S}(T) \) in (a) is also satisfied. Otherwise, \( T \) is *ambiguous*. 

of unambiguous events for \((u, \nu)\). The set of axioms that they impose on the preference relation guarantees that the set of ‘unambiguous’ events is rich enough so that ‘beliefs’ over these events can be represented by a countably additive, convex-valued probability measure 
\(\pi : \mathcal{E}^U \rightarrow [0, 1]\). Moreover, for each \(A \in \mathcal{E}^U\), \(\nu(A) = \phi\left(\pi(A)\right)\), for some strictly increasing and onto map \(\phi : [0, 1] \rightarrow [0, 1]\). Hence for any (measurable) finite partition, \((A^1, \ldots, A^n)\), and for all acts of the form \(f = x^1_{A_1}x^2_{A_2} \ldots x^{n-1}_{A_{n-1}}x^n\) for which \(x^1 \geq \ldots \geq x^n\), the certainty equivalent function, \(m(f)\) for the CEU maximizer, \((u, \nu)\), is defined by:

\[m(f) = u^{-1}\left(\sum_{i=1}^{n} \left( u\left(x^i\right)\nu\left(\bigcup_{j=1}^{i}A^j\right) - u\left(x^i\right)\nu\left(\bigcup_{j=1}^{i-1}A^j\right)\right)\right).\]

Furthermore, if for each \(i = 1, \ldots, n\), \(A^i \in \mathcal{E}^{U}\), then \(f\) is an unambiguous act and

\[m(f) = u^{-1}\left(\sum_{i=1}^{n} \left( u\left(x^i\right)\phi\left(\pi\left(\bigcup_{j=1}^{i}A^j\right)\right) - u\left(x^i\right)\phi\left(\pi\left(\bigcup_{j=1}^{i-1}A^j\right)\right)\right)\right).\]

If we take another such CEU maximizer \((u, \hat{\nu})\), for whom \(\mathcal{E}^U_{\hat{\nu}} = \mathcal{E}\) (that is, every event is unambiguous for this individual) and \(\hat{\nu}(A) = \nu(A)\) for every \(A \in \mathcal{E}^U_{\nu}\), then by construction the two CEU maximizers, \((u, \nu)\) and \((u, \hat{\nu})\), agree over any pair of acts that are measurable with respect to \(\mathcal{E}^U_{\nu}\). Furthermore, since every event is unambiguous for \((u, \hat{\nu})\), this CEU-maximizer is probabilistically sophisticated in the sense of Machina and Schmeidler (1992), and so corresponds to Epstein’s (1999) notion of an ambiguity neutral preference relation. Thus there exists a countably additive, convex-valued probability measure, \(\hat{\pi}\) that extends \(\pi\) to \(\mathcal{E}\). That is, for any \(E \in \mathcal{E}\), \(\hat{\nu}(E) = \phi\left(\hat{\pi}(E)\right)\), and for any (measurable) finite partition, \((A^1, \ldots, A^n)\), and for all acts of the form \(f = x^1_{A_1}x^2_{A_2} \ldots x^{n-1}_{A_{n-1}}x^n\) for which \(x^1 \geq \ldots \geq x^n\), the certainty equivalent function, \(\hat{m}(f)\) for the CEU maximizer, \((u, \hat{\nu})\), is defined by:

\[
\hat{m}(f) = u^{-1}\left(\sum_{i=1}^{n} \left( u\left(x^i\right)\hat{\nu}\left(\bigcup_{j=1}^{i}A^j\right) - u\left(x^i\right)\hat{\nu}\left(\bigcup_{j=1}^{i-1}A^j\right)\right)\right)
\]

According to Epstein’s (1999) definition, \((u, \nu)\) is ambiguity averse if for any pair of acts \(f\) and \(h\), such that \(h\) is measurable with respect to \(\mathcal{E}^U_{\nu}\),

\[
\hat{m}(h) \geq \hat{m}(f) \text{ implies } m(h) \geq m(f).
\]

Epstein and Zhang (2001) show that \((u, \nu)\) is ambiguity averse if and only if

\[
\hat{\pi}(E) \geq \phi^{-1}(\nu(E)) \text{ for all } E \in \mathcal{E}.
\]
The following corollary to Proposition (9) establishes the connection of our definition of more uncertainty averse to Epstein’s (1999) definition of ambiguity averse.

**Corollary 11** Let $\succeq$ and $\tilde{\succeq}$ be preference relations corresponding to the CEU-maximizers $(u, \nu)$ and $(\tilde{u}, \tilde{\nu})$ defined above. Then $(u, \nu)$ is ambiguity averse in the sense of Epstein (1999), if and only if $\succeq$ is more uncertainty averse than $\tilde{\succeq}$.

From this corollary we can conclude that a CEU-maximizer is ambiguity averse in the sense of Epstein (1999) if and only if there is a probabilistically sophisticated CEU-maximizer, such that: (a) the two preference relations agree over the set of unambiguous acts; and (b) the former is more uncertainty averse than the latter.

To explore what risk aversion may entail in this setting, consider a third CEU maximizer $(\tilde{u}, \tilde{\nu})$, where $\tilde{u}(x) = x$ for all $x \in \mathbb{R}_+$. This individual is actually a subjective-expected-value maximizer whose beliefs about the likelihood of events agrees with the CEU-maximizer $(u, \nu)$ for all events in $\mathcal{E}^{UA}_\nu$. Using the probability measure $p$ defined over the events in $\mathcal{E}^{UA}_\nu$, we can form the partial ordering of second order stochastic dominance over the set of acts measurable with respect to $\mathcal{E}^{UA}_\nu$, as follows:

**Definition 5** Fix a probability measure $\pi$ defined over the events in $\mathcal{E}^{UA}_\nu$. For any finite partition $(A^1, \ldots, A^n)$, such that $A^i \in \mathcal{E}^{UA}_\nu$, for all $i = 1, \ldots, n$, and any pair of acts $f = x^1_{A^1} \ldots x^{n-1}_{A^{n-1}} x^n$ and $g = y^1_{A^1} \ldots y^{n-1}_{A^{n-1}} y^n$ where $x^1 \geq \ldots \geq x^n$ and $y^1 \geq \ldots \geq y^n$, we say $f$ second order stochastically dominates $g$ (with respect to $\pi$) if

$$
\sum_{j=1}^{n} (x^j - y^j) \pi(A^j) \geq 0, \text{ for all } i = 1, \ldots, n.
$$

As is well-known, the CEU-maximizer $(u, \nu)$ agrees with the partial ordering of second order stochastic dominance over the set of acts that are measurable with respect to $\mathcal{E}^{UA}_\nu$, if and only if $u$ is concave and $\phi$ is convex (see for example, Chew, Karni and Safra, 1987).\(^6\)

There are, however, a number of weaker notions of risk aversion. We shall consider two.\(^7\)

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\(^6\) For a characterization of aversion to mean-preserving spreads in the cumulative prospect theory model see Schmidt and Zank (2002).

\(^7\) The interested reader is referred to the excellent overview by Chateauneuf, Cohen and Meilijson (2001) that provides a taxonomy of five distinct characterizations of risk aversion for models of decision making under risk.
Definition 6 Fix a probability measure $\pi$ defined over the events in $\mathcal{E}_U^{UA}$. The preference relation $\succcurlyeq$ defined over the set of acts that are measurable with respect to $\mathcal{E}_U^{UA}$, is weakly risk averse (with respect to the probability measure $\pi$) if for any finite partition $(A^1, \ldots, A^n)$, such that $A^i \in \mathcal{E}_U^{UA}$, for all $i = 1, \ldots, n$, and any act $f = x_{A^1} \cdots x_{A^{n-1}} x^n: \left(\sum_{i=1}^n \pi(A^i) x^i\right)_S \succcurlyeq f$.

There is no known characterization of weak risk aversion for a probabilistically sophisticated CEU-maximizer. Chateauneuf and Cohen (1994), do provide, however, sufficient conditions that do not imply the concavity of $u$. The second alternative, aversion to monotone mean-preserving spreads, in terms of its strength, lies between the weak definition and the one based on the second-order stochastic dominance relation.

Definition 7 Fix a probability measure $\pi$ defined over the events in $\mathcal{E}_U^{UA}$. For any finite partitions $(A^1, \ldots, A^n)$ and $(B^1, \ldots, B^n)$ such that $A^i, B^i \in \mathcal{E}_U^{UA}$, and $\pi(A^i) = \pi(B^i)$, for all $i = 1, \ldots, n$, and any pair of acts $f = x_{A^1} \cdots x_{A^{n-1}} x^n$ and $g = y_{B^1} \cdots y_{B^{n-1}} y^n$ where $x^1 \geq \ldots \geq x^n$ and $y^1 \geq \ldots \geq y^n$, we say $g$ is a monotone mean-preserving increase in risk (with respect to $\pi$) of $f$ if

$$
\left(\pi(A^i) x^i - \pi(A^j) y^j\right) (i - j) \geq 0 \text{ for all } i, j \in \{1, \ldots, n\}
$$

and $\sum_{i=1}^n \left(\pi(A^i) x^i - \pi(A^j) y^j\right) = 0$.

A preference relation $\succcurlyeq$ defined over acts that are measurable with respect to $\mathcal{E}_U^{UA}$ is said to be averse to monotone mean-preserving increases in risk, if for any such pair of (unambiguous) acts $g$ and $f$ as defined above, $f \succcurlyeq g$.

In the context of decision making under risk, where preferences are defined over lotteries, Chateauneuf, Cohen and Meilijson (1997) provide a complete characterization for a Rank-Dependent Expected Utility maximizer to be averse to all monotone mean preserving increases in risk. Adapting their result to the subjectively uncertain act-framework here, we have that the CEU-maximizer $(u, \nu)$ is (weakly) averse to any monotone mean-preserving increase in risk (with respect to $\pi$) if and only if

$$
\inf_{E \in \mathcal{E}_U^{UA}} \left[ \frac{(1 - \phi(\pi(E))) \phi(\pi(E))}{(1 - \pi(E)) \pi(E)} \right] \geq \sup_{x \geq y} \frac{u'(x)}{u'(y)}.
$$

---

8 One implication of Chateauneuf and Cohen’s (1994) result is that the claim by Epstein and Zhang (2001, p287) in Corollary 7.4 (a) that the concavity of $u$ is necessary for a CEU-maximizer to be risk-averse (in the weak sense) over acts that are measurable with respect to the set of unambiguous acts is incorrect.
They refer to the left-hand expression of (5) as the index of \textit{pessimism} of $\phi$. They dub the right-hand expression as the index of \textit{greediness} and note that if it equals one, then $u$ is concave. From (5), it follows that a necessary requirement for $\phi$ to satisfy is that

$$\inf_{E \in \mathcal{E}^{UA}_\nu} \left[ \frac{(1 - \phi(\pi(E)))}{\phi(\pi(E))} \right] \geq 1$$

that is, $\phi(q) \leq q$, for all $q \in [0, 1]$. However, as they emphasize, the most significant feature of (5) is that $u$ need not be concave, for the CEU-maximizer $(u, \nu)$ to be monotone risk averse.

A hint at the connection between this notion of risk aversion and our model-free definition of increases in uncertainty, is suggested by the fact that if $g$ is a monotone mean preserving increase in risk (with respect to $\pi$) of $f$ and $A^i = B^i$, for all $i = 1, \ldots, n$, then $g \overleftarrow{U} f$. And indeed, the following corollary of Chateauneuf, Cohen and Meilijson’s (1997) characterization result shows that the relationship is tight.

\textbf{Corollary 12} Let $\succsim$ and $\prec\succsim$ be preference relations defined over acts that are measurable with respect to $\mathcal{E}^{UA}_\nu$, and that agree on this restricted domain, with the CEU-maximizers $(u, \nu)$ and $(\tilde{u}, \tilde{\nu})$, respectively. Then $\succsim$ is at least as uncertainty averse as $\prec\succsim$ on this restricted domain if and only if (5) is satisfied.

\section{Alternative notions of elementary increases in risk and uncertainty}

The definition of an elementary increase in uncertainty presented above is the simplest possible. As we have shown, its transitive closure is a monotone spread. Corollary 12 demonstrates that this result also holds under risk (see also Quiggin, 1993). There are, however, a wide range of alternative notions of increasing risk. Chateauneuf, Cohen and Meilijson (2002) give a summary, and their discussion suggests a systematic procedure for generating various classes of increases in risk as the transitive closure of appropriate notions of an elementary increase in risk. In addition to monotone spreads, Chateauneuf, Cohen and Meilijson consider the Rothschild-Stiglitz mean-preserving riskier ordering and two intermediate orderings, referred to as left-monotone and right-monotone increases in risk.

In the case of risk, these intermediate orderings can be generated as transitive closures of elementary increases in risk based on the following notion of three-event and four-event \textit{ordered} partitions of the state space.
Definition 8 Fix an act \( f \). The \( N \)-event partition \( \{E^1, \ldots, E^N\} \) of \( S \) is ordered with respect to \( f \) if for all \( n = 1, \ldots, N - 1 \)
\[
\sup \{f(s) : s \in E^n\} \leq \inf \{f(s) : s \in E^{n-1}\}.
\]

Definition 9 An act \( g \) is a elementary left-increase in uncertainty on \( f \) if there exist numbers \( \alpha \) and \( \beta \) and a 3-event partition \( \{E^1, E^2, E^3\} \) that is ordered with respect to \( f \) and \( g \), such that
\[
g(s) = \begin{cases} 
  f(s) - \beta & s \in E^1 \\
  f(s) + \alpha & s \in E^2 \\
  f(s) & s \in E^3
\end{cases}
\]

Definition 10 An act \( g \) is a elementary right-increase in uncertainty on \( f \) if there exist numbers \( \alpha \) and \( \beta \) and a 3-event partition \( \{E^1, E^2, E^3\} \) that is ordered with respect to \( f \) and \( g \), such that
\[
g(s) = \begin{cases} 
  f(s) & s \in E^1 \\
  f(s) - \beta & s \in E^2 \\
  f(s) + \alpha & s \in E^3
\end{cases}
\]

Definition 11 An act \( g \) is a monotone increase in uncertainty on \( f \) if there exist numbers \( \alpha \) and \( \beta \) and a 3-event partition \( \{E^1, E^2, E^3\} \) that is ordered with respect to \( f \) such that
\[
g(s) = \begin{cases} 
  f(s) - \beta & s \in E^1 \\
  f(s) & s \in E^2 \\
  f(s) + \alpha & s \in E^3
\end{cases}
\]

As the terminology suggests, if \( g \) is a monotone increase in uncertainty on \( f \), then \( g \) is more uncertain than \( f \) in the sense of Definition 3. Indeed, for any \( \gamma < \min(\alpha, \beta) \), we can generate a monotone increase in uncertainty from the sequence of elementary increases in uncertainty, \( h^1 = f \), \( h^2(s) = f(s) - \beta + \gamma \), \( s \in E^1 \), \( h^2(s) = f(s) + \gamma \), \( s \in E^2 \cup E^3 \), \( h^3 = g \), that is \( h^3(s) = h^2(s) - \gamma \), \( s \in E^1 \cup E^2 \), \( h^2(s) = h^3(s) + \alpha - \gamma \), \( s \in E^3 \).

Using the results of Chateauneuf, Cohen and Meilijson (Lemma 2, p11), it is straightforward to show that, if we assume known probabilities, and add the requirement that \( g \) has the same expected value as \( f \), the transitive closure of the class of elementary left-(respectively, right-)increases in uncertainty on \( f \) is the class of acts that are ranked left-(respectively, right-)monotone more riskier than \( f \). Also observe that an elementary increase in uncertainty satisfies each of the definitions 9, 10 and 11 with the ‘unchanged’ event being empty.

Now consider potential notions of elementary increases in uncertainty of an act with respect to a four-event ordered partition. The only elementary operation that is (i) measurable with respect to a four-event partition, (ii) does not include an increase on \( E^1 \) and a
reduction in $E^j$, for all $j > i$, and (iii) cannot be generated by a finite sequence of any of the elementary increases considered above, is the following.

**Definition 12** An act $g$ is an elementary conditional-increase in uncertainty on $f$ if there exist numbers $\alpha$ and $\beta$ and a 4-event partition \( \{E^1, E^2, E^3, E^4\} \) that is ordered with respect to $f$ and $g$, such that

\[
g(s) = \begin{cases} 
  f(s) & s \in E^1 \\
  f(s) - \beta & s \in E^2 \\
  f(s) + \alpha & s \in E^3 \\
  f(s) & s \in E^4 
\end{cases}
\]

Observe that elementary, left-elementary and right-elementary increases in uncertainty are all special cases of elementary conditional-increase in uncertainty. Moreover, if we assume known probabilities, and add the requirement $E[g] = E[f]$, the transitive closure of the relation ‘$g$ is derived from $f$’ by an elementary conditional-increase in uncertainty’ is the relation ‘$g$ is mean-preserving riskier than $f$’ in the sense of Rothschild-Stiglitz.

Our main result in this section is that no such relationship applies under uncertainty. In fact, if we suppose that the state-space is sufficiently rich, in the sense that there are no “atoms” in the state space, then it follows that the transitive closure is the trivial total ordering, which includes every ordered pair of acts. That is, suppose we require in addition to the other maintained assumptions that the preference relation satisfies Savage’s postulate small-event continuity.

**Definition 13** The relation $\succsim$ exhibits small event continuity if for any pair of acts $f \succ g$, and any outcome $x$, there exists a finite partition of the state space \( \{E^1, \ldots, E^N\} \) such that $x_{E^n}f \succ g$ and $f \succ x_{E^n}g$ for every $n = 1, \ldots, N$.

**Proposition 13** Suppose $\succsim$ exhibits small event continuity. The transitive closure of the relation ‘$g$ is derived from $f$ by an elementary conditional-increase in uncertainty’ is the full relation $R$ in which, for all pairs of act $f$ and $g$, $gRf$ and $fRg$.

Thus, there is no non-trivial analog under uncertainty for the Rothschild-Stiglitz notion of an increase in risk.
5 Economic Implications of Increasing Uncertainty Aversion

Choice sets in which all (undominated) elements are ordered by $\bar{\mathcal{U}}$ arise naturally in a range of economic problems. In the one safe asset, one risky asset portfolio problem, analyzed by Pratt (1964) and many others, $g\bar{\mathcal{U}}f$ whenever $g$ represents a portfolio with more of the risky asset. More generally, standard capital market theory implies that idiosyncratic risk can be completely traded away. Hence, net of such trades, all financial assets may be assumed to be comonotonic with the market portfolio. If the two fund separation property applies, and the difference between the state-contingent returns to the two funds are increasing and display a single-crossing property, then any portfolio may be represented as a mixture of two acts $f$ and $g$ with $g\bar{\mathcal{U}}f$, and where the (normalized) prices of both acts are 1. We may also allow for fixed background risk $h$, representing, say, (normalized) returns to human capital, with the assumption that $(f + h)\bar{\mathcal{U}}f$. This will be true if and only if $h$ is comonotonic with $f$ and takes both positive and negative values. Hence, we also have $(g + h)\bar{\mathcal{U}}g$. Thus, the problem of an individual deciding how to allocate her initial wealth of $w$, is to find the optimal holding $k$ of the more uncertain asset, and the associated optimal contingent income $f^*$, where

$$f^* \in \mathcal{B} \equiv \left\{ \tilde{f} \in \mathcal{F} : \tilde{f} = h + kg + (w - k)g \text{ for some } k \in [0, 1] \right\}$$

s.t. $f^* \succ \tilde{f}$, for all $\tilde{f} \in \mathcal{B}$.

The one safe asset, one risky asset portfolio problem arises when $h(s) \equiv 0$ and $f$ is a constant act.

We immediately obtain:

**Corollary 14** Let $\succ$ be everywhere at least as uncertainty averse as $\bar{\succ}$ and let $f^*$ (respectively, $\hat{f}^*$) be the optimal contingent income for $\succ$ (respectively, $\hat{\succ}$) and $k^*$ (respectively, $\hat{k}^*$) the associated allocation to the less uncertain asset. Then $\hat{f}^* \bar{\mathcal{U}} f^*$, that is, $k^* \leq \hat{k}^*$.

Similarly, the standard comparative static results derived by Pratt (1964) and subsequent writers apply with no requirement for well-defined probabilities. In particular, consider the following analog of constant absolute risk aversion.

**Definition 14** Fix a preference relation $\succ$ and let $m()$ be the associated certainty equivalent function. We say $\succ$ displays constant (decreasing, increasing) absolute uncertainty aversion if for all acts $f \in \mathcal{F}$ and outcomes $x \in \mathbb{R}_+$, $m(f + x) = (\geq, \leq) m(f) + x$. 

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Corollary 15 Let $\succsim$ display decreasing absolute uncertainty aversion. Then either (i) an increase in wealth $w$; or (ii) a reduction in the price of the more uncertain asset will lead to an increase in the optimal holding $k$.

The portfolio allocation problem is the canonical application of the theory of choice under risk, in part because there is a great deal of data on which to base judgements about relative probabilities. An application in which the problem of ambiguity is more salient is that of individual or collective self-protection against environmental hazards (Shogren and Crocker 1991, Quiggin 1992, 2002). Consider a state set $\mathcal{S} = \{\ldots, s, \ldots\}$ ordered in such a way that higher values of $s$ correspond to lower levels of some environmental hazard. A commonly used example for individual choice is that of atmospheric concentrations of lead. Similar problems involving collective choice arise with respect to global pollutants such as chlorofluorocarbons (CFCs) and greenhouse gases, where there exists considerable nonprobabilistic uncertainty about growth in atmospheric concentrations and about the associated climatic consequences and feedbacks.

Quiggin (2002), analysing choice under risk, derives bounds on willingness to pay for reductions in ambient hazard, based on the existence of a self-protection technology. To extend these results to general problems involving uncertainty, it is only necessary to generalize the definitions of a self-protection technology and of a reduction in ambient hazard. Following Quiggin (2002) we represent the ambient hazard state by $d \in \mathbb{R}$ and the hazard-mitigation resources committed by the individual by $e \in \mathbb{R}_+$. Let health outcomes be represented by a scalar ‘quality-of-life’, so that the individual’s state-contingent health outcome may be represented by a vector $q \in \mathbb{R}^S$. For given ambient hazard $d$ and commitment of resources $e$ the individual’s choice set is taken to be

$$Y(d) = \{ (q,e) \in \mathbb{R}^S \times \mathbb{R}_+: (q,e) \text{ is feasible given ambient hazard } d \}.$$ 

As noted above, it is assumed that both health outcomes and hazard-mitigation efforts may be valued in monetary terms, and further that welfare is additively separable in consumption and health status. Then, given effort $e$ and health vector $q$, the individual’s \textit{ex post} equivalent wealth in state $s$ is

$$f(s; q, p, e) = \eta(q(s)) - pe,$$

where $\eta$ represents the monetary equivalent of health status $q$, and $p \in \mathbb{R}_{++}$ is an input price for hazard-mitigation effort. We assume that higher values of $q$ are better so that $\eta$ is monotonically increasing. We may regard the individual’s problem as one of choosing
$(q,e) \in Y(d)$ to maximize some objective function $V(f)$ where $V : \mathbb{R}_+^n \to \mathbb{R}$ is assumed to be quasi-concave and increasing. Let $q^*(p,d)$ and $e^*(p,d)$ denote (selections) from the solution of this constrained maximization problem.

For any given state-contingent distribution of health outcomes $q$, define the cost function

$$C(q;p,d) = \min_{(e \in \mathbb{R}_+)} \{pe : (q,e) \in Y(d)\}.$$ 

and let

$$e^*(p,d) = C(q^*(p,d);p,d) = pe^*(p,d)$$

Further define

$$q^*(p,d,c) = \arg \max \{ V(f) : C(q;p,d) \leq c \}$$

Chambers and Quiggin (2000) show that, if the cost function $C$ is homothetic, the state space will have a natural ordering $\leq$ in the sense that, for all $c$ and $s \leq s'$,

$$\eta(q^*(s;p,d,c)) \leq \eta(q^*(s';p,d,c))$$  \hspace{1cm} (6)

Note that states are ordered from worst to best. For example, suppose that $s$ denotes the ambient concentration of some pollutant. Then low values of $s$ correspond to high levels of pollution.

Condition (6) requires that, under an optimal self-protection plan, more health damage is sustained on high-pollution days than on low-pollution days. This need not be the case. For example, a self-protection plan in which responses to pollution were undertaken only when the ambient concentration exceeds some ‘trigger’ level, which might be conditional on cost, would typically incur less health damage on days when ambient concentrations are just above the trigger level than on days when pollution is just below. However, if the cost function is homothetic, such a plan will not be optimal.

A stronger condition is that, for all $p$, whenever $s \leq s'$ and $c \leq c'$,

$$\eta(q^*(s;p,d,c')) - \eta(q^*(s;p,d,c)) \geq \eta(q^*(s';p,d,c')) - \eta(q^*(s';p,d,c))$$  \hspace{1cm} (7)

That is, an increase in expenditure produces a greater reduction in damage, the worse is the state of nature. Consider again the case when $s$ indexes the ambient concentration of a pollutant, and suppose that the production technology involves filtering pollutants. Condition (7) will be satisfied if the proportion of pollutants filtered is an increasing function of the ambient concentration and damage is a convex function of exposure to pollution after filtering. It implies that, for all $d, p, c \leq c', f^*(p,d,c) \bar{U} f^*(p,d,c')$ where
\[ f^*(s; p, d, c) = \eta(\tilde{q}^*(s; p, d, c)) - c \]

The willingness to pay for ambient hazard reduction relative to some base level of ambient conditions \(d^0\) is given by

\[ WTP(d) = \max\{\delta : \max_{(q, e) \in Y(d)} V(\eta(q) - (pe + \delta) \mathbf{1}) \leq \max_{(q, e) \in Y(d^0)} V(\eta(q) - (pe) \mathbf{1})\}. \]

where \(\mathbf{1} \in \mathbb{R}^S\) is the vector with all entries equal to 1.

Using the arguments of Quiggin (2002), it is straightforward to show that willingness to pay for a change in ambient hazard from \(d^0\) to \(d^1\) is bounded by

\[ C(q^*(p, d^0); d^0) - C(q^*(p, d^0); d^1) \leq WTP(d^1; p) \leq C(q^*(p, d^1); d^0) - C(q^*(p, d^1); d^1). \]

If \(C\) is differentiable in \(q\) and \(d\), we obtain by a straightforward application of the envelope theorem:

\[ \frac{\partial WTP(d; p)}{\partial d} = \frac{\partial C(q^*(p, d); d)}{\partial d}. \quad (8) \]

This is an extension of a result originally derived under certainty by Bartik (1988).

Now define a reduction in ambient hazard as a shift from \(d^0\) to \(d^1\) such that

(a) for all \(q\),

\[ C(q; p, d^1) \leq C(q; p, d^0) \]

(b) for all \(q, q'\) where \(\eta(q(s))\) is an increase in uncertainty on \(\eta(q'(s))\)

\[ C(q'; p, d^1) - C(q; p, d^1) \leq C(q'; p, d^1) - C(q; p, d^1) \]

Describe two preference orderings \(\succeq\) and \(\widehat{\succeq}\) as comparable if, for all \(c, d,\)

\[ q^*(p, d, c) = \tilde{q}^*(p, d, c) \]

For any such pair of comparable preference relations, we can show that for a given level of the ambient hazard, the optimal expenditure on hazard mitigating resources is no lower for the more uncertainty averse preference relation. Furthermore, the willingness to pay for any reduction in the ambient hazard is no lower for the more uncertainty averse preference relation. More precisely, we have:
Proposition 16 Assume conditions (6) and (7) hold. Suppose $\succeq$ and $\tilde{\succeq}$ are comparable and $\succeq$ is everywhere at least as uncertainty averse as $\tilde{\succeq}$. Then

1. For any $d$, $c^*(p, d) \geq \tilde{c}^*(p, d)$

2. If $C$ is everywhere differentiable, then for any reduction in ambient hazard from $d_0$ to $d_1$

$$WTP(d_1; p) \geq \tilde{WTP}(d_1; p)$$

We can also obtain a range of comparative static results. The assumption of constant absolute risk aversion is particularly helpful, since it means that wealth effects can be disregarded. In particular, recall that

$$q^*(p_0, d, c) = \arg \max \{ V(f) : C(q; p^0, d) \leq c \}$$

Since the cost function is homogeneous of degree 1 in $p$, observe that

$$\{ q : C(q; p^0, d) \leq c \} = \left\{ q : C(q; p^1, d) \leq \frac{cp^1}{p^0} \right\}$$

Now, suppose $C(q; p^0, d) = C(q; p^0, d) = c$, and let the associated distributions be denoted $f, \hat{f}$. Under CARA, if $m(f) \geq m\left(\hat{f}\right), m\left(\hat{f} + \frac{cp^1}{p^0} - c\right) \geq m\left(\hat{f} + \frac{cp^1}{p^0} - c\right)$. It follows that

$$q^*(p^1, d, \frac{cp^1}{p^0}) = q^*(p^0, d, c)$$

This result permits us to derive:

Proposition 17 If preferences $\succeq$ display constant absolute uncertainty aversion, an increase in $p$ leads to a reduction in $q$.

6 Conclusion

Most economic analysis of choice under uncertainty, and particularly of increases in uncertainty, has been based on the assumption that decision-makers have well-defined subjective probabilities. On the other hand, the fundamental result of the literature, the proof of existence of equilibrium in state-contingent markets derived by Arrow and Debreu (1954), does not require decision-makers to possess subjective probabilities or to satisfy the postulates of any model specific to problems involving uncertainty.
In this paper, we have shown that some, but not all, of the concepts that have been used in the case of known probabilities can be extended to the more general and realistic case of unknown probabilities. Broadly speaking, concepts that are most naturally expressed in state-continent terms, such as statewise dominance and monotone spreads, are robust. Concepts that are most naturally expressed in terms probabilities or cumulative probability distributions, such as notions of stochastic dominance, are unlikely to be robust.

Appendix

Proof of Proposition 3.

We first establish the following lemmas.

Lemma 18 If for any pair of simple acts $g$ and $f$, any pair of positive real numbers, $\alpha$ and $\beta$, and any three element partition of $S$, $(E_{\alpha}, E_0, E_{\beta})$, we have

$$g(s) - f(s) = \begin{cases} 
\alpha & \text{if } s \in E_1 \\
0 & \text{if } s \in E_0 \\
-\beta & \text{if } s \in E_{\beta} 
\end{cases}$$

then there exists a simple act $h$ for which $gU_h$ and $hU_f$.

Proof: If $\alpha > \beta/2$ then define

$$h(s) = \begin{cases} 
\frac{f(s) + \alpha - \beta}{2} & \text{if } s \in E_1 \\
\frac{f(s) - \beta}{2} & \text{if } s \in E_0 \\
\frac{f(s) - \beta}{2} & \text{if } s \in E_{\beta}. 
\end{cases}$$

Notice that

$$g(s) - h(s) = \begin{cases} 
\frac{\beta}{2} & \text{if } s \in E_1 \cup E_0 \\
-\beta/2 & \text{if } s \in E_{\beta} 
\end{cases}$$

and

$$h(s) - f(s) = \begin{cases} 
\frac{\alpha - \beta}{2} & \text{if } s \in E_1 \\
-\beta/2 & \text{if } s \in E_0 \cup E_{\beta} 
\end{cases}$$

as required. If $\alpha \leq \beta/2$ then define

$$h(s) = \begin{cases} 
\frac{f(s) + \alpha}{2} & \text{if } s \in E_1 \\
\frac{f(s) + \alpha}{2} & \text{if } s \in E_0 \\
\frac{f(s) - \beta + \alpha}{2} & \text{if } s \in E_{\beta}. 
\end{cases}$$
Now we have
\[
g(s) - h(s) = \begin{cases} 
\alpha/2 & \text{if } s \in E_1 \\
-\alpha/2 & \text{if } s \in E_0 \cup E_{-1}
\end{cases}
\quad \text{and } h(s) - f(s) = \begin{cases} 
\alpha/2 & \text{if } s \in E_1 \cup E_0 \\
-\beta + \alpha/2 & \text{if } s \in E_{-1}.
\end{cases}
\]

\[\blacksquare\]

**Lemma 19** If for any pair of simple acts \(f\) and \(g\), \(gU f\) then there exists a finite sequence of simple acts \((h_m)_{m=1}^M\) such that \(h_1 = f\), \(h_M = g\) and \(h_{m+1}Uh_m\), \(m = 1, \ldots, M - 1\).

**Proof:** From the definition of \(gU f\) it follows that \(g - f\) is pairwise co-monotonic with both \(g\) and \(f\). Let \([E_{-J}, \ldots, E_1, E_0, E_1, \ldots, E_I]\) be the coarsest partition of \(S\) for which \(g - f\) is measurable and with the labelling monotonically ordered, that is for any \(i > j\), and any \(s \in E_i\) and \(s' \in E_j\), \(g(s) - f(s) > g(s') - f(s')\). Moreover, assume that for any \(i < 0\), and any \(s \in E_i\), \(g(s) - f(s) < 0\), for any \(i > 0\) and any \(s \in E_i\), \(g(s) - f(s) > 0\); and for any \(s \in E_0\), \(g(s) = f(s)\) \(E_0\) may be empty, but since \(\inf_{s \in S} g(s) < \inf_{s \in S} f(s)\) and \(\sup_{s \in S} g(s) > \sup_{s \in S} f(s)\) it follows that \(I \geq 1\) and \(J \geq 1\). For each \(i = -J, \ldots, 0, \ldots, I\), and some \(s_i \in E_i\), set \(d_i := g(s_i) - f(s_i)\). By construction, we have
\[
d_{-J} < d_{-J+1} < \ldots < d_{-1} < d_0 = 0 < d_1 < \ldots < d_I.
\]

Let \(h_1 := f\). Define
\[
h_2(s) = \begin{cases} 
f(s) + d_1 & \text{if } s \in E_1 \cup E_2 \cup \ldots \cup E_I \\
f(s) & \text{if } s \in E_0 \\
f(s) + d_{-1} & \text{if } s \in E_{-1} \cup E_{-2} \cup \ldots \cup E_{-J}.
\end{cases}
\]

For \(i = 2, \ldots, \min\{I, J\} - 1\), define
\[
h_{2i+1}(s) = \begin{cases} 
h_{2i-1}(s) + d_i - d_{i-1} & \text{if } s \in E_{i} \cup \ldots \cup E_I \\
h_{2i-1}(s) & \text{if } s \in E_{-i+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\
h_{2i-1}(s) + d_{-i} - d_{-i+1} & \text{if } s \in E_{-i} \cup \ldots \cup E_{-J}.
\end{cases}
\]

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\[ I \geq J, \text{ then for } i = J, \ldots, I, \text{ define} \]
\[
\begin{align*}
  h_{2i+1}(s) = \begin{cases} 
  h_{2i-1}(s) + d_i - d_{i-1} & \text{if } s \in E_i \cup \ldots \cup E_I \\
  h_{2i-1}(s) & \text{if } s \in E_{-J+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\
  h_{2i-1}(s) + (d_{-J} - d_{-J+1}) / (I - J + 1) & \text{if } s \in E_{-J}.
\end{cases}
\end{align*}
\]

Notice, in this case \( h_{2I+1} = g. \)

If, however, \( I < J, \) then for \( i = I, \ldots, J, \) define
\[
\begin{align*}
  h_{2i+1}(s) = \begin{cases} 
  h_{2i-1}(s) + (d_i - d_{i-1}) / (J - I + 1) & \text{if } s \in E_I \\
  h_{2i-1}(s) & \text{if } s \in E_{-i+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\
  h_{2i-1}(s) + d_{-i} - d_{-i+1} & \text{if } s \in E_{-i} \cup \ldots \cup E_{-J}
\end{cases}
\end{align*}
\]

and now \( h_{2J+1} = g. \)

For each \( i = 1, \ldots, \max \{I, J\}, \) it follows from Lemma (18) that there exists a simple act \( h_{2i} \) for which \( h_{2i+1} U h_{2i} \) and \( h_{2i} U h_{2i-1}. \) Hence we have
\[ g = h_{2 \max \{I, J\} + 1} U h_{2 \max \{I, J\}} U \ldots U h_1 = f \]
as required.

We are now in a position to prove the proposition. Let \( f_n \) and \( h_n \) be the usual uniform simple approximations from below of \( f \) and \( h; \) for \( n \) large enough \( \sup h_n > 0 \) and \( \inf h_n < 0. \) Moreover, as noted in Chateauneuf, Cohen and Meiljison (1997), \( f \) and \( h \) comonotonic implies that \( f_n \) and \( h_n \) are comonotonic, and therefore result follows from Lemma (19). \( \Box \)

**Proof of Proposition 7.**

Sufficiency is obvious. For necessity of the equality of \( \pi \) and \( \hat{\pi}, \) consider choices in a neighborhood of a constant act \( x. \) For any real-valued function \( d : \mathcal{S} \to \mathbb{R} \) and sufficiently small \( \varepsilon > 0, \) the certainty equivalent of the act \( x + \varepsilon d \) (in the neighborhood of \( x \)) under \( m \) is approximately
\[ x + \varepsilon \int d(s) \pi (ds) \]
and, similarly for \( \hat{m}, \) the certainty equivalent is
\[ x + \varepsilon \int d(s) \hat{\pi} (ds). \]
If \( \pi (E) > \hat{\pi} (E) \) for some \( E \subset \mathcal{S} \) then
\[
\frac{\pi (E)}{1 - \pi (E)} > \frac{\hat{\pi} (E)}{1 - \hat{\pi} (E)} - \frac{\delta}{(1 - \pi (E))(1 - \hat{\pi} (E))}
\]
for some \( \delta > 0 \). Thus if we take
\[
d(s) = \begin{cases} 
1 - \hat{\pi} (E) - \delta & \text{if } s \in E \\
-\hat{\pi} (E) - \delta & \text{if } s \notin E,
\end{cases}
\]
then for any \( \varepsilon > 0 \) we have
\[
\varepsilon \int_s d(s) \pi (ds) > 0 > \varepsilon \int_s d(s) \hat{\pi} (ds)
\]
and \( x + \varepsilon d \mathcal{U} x \). So for sufficiently small \( \varepsilon > 0 \), it follows from continuity and monotonicity of \( \succeq \) and \( \sim \), that \( x \sim (x + \varepsilon d) \) but \( (x + \varepsilon d) \succ x \).

To demonstrate the necessity of \( u \) being a concave transformation of \( \hat{u} \), suppose the contrary, that is, \( u \) is not a concave transform of \( \hat{u} \). Then there must exist utility levels \( v_1, v_2 \) and \( v_3 \) in the range of \( \hat{u} \), and \( \lambda \) in \( (0, 1) \), such that
\[
\lambda v_1 + (1 - \lambda) v_3 = v_2
\]
\[
\lambda u \circ \hat{u}^{-1} (v_1) + (1 - \lambda) u \circ \hat{u}^{-1} (v_3) > u \circ \hat{u}^{-1} (v_2).
\]

Since \( \pi \) is non-atomic, there exists an event \( E \subset \mathcal{S} \) for which \( \pi (E) = \lambda \). So consider the act
\( f := \hat{u}^{-1} (v_1) \circ \hat{u}^{-1} (v_3) \) and the constant act \( x := \hat{u}^{-1} (v_2) \). By construction we have \( f \succ x \), \( x \succeq f \) and \( f \succ x \).

**Proof of Proposition 8.**

Proof: \( (3) \Rightarrow (2) \Rightarrow (1) \) is straightforward

Hence, we need only prove \( (1) \Rightarrow (3) \). The proof is in two parts. We first show that \( (1) \) requires \( \pi (A) = \hat{\pi} (A) \) for all \( A \in \mathcal{E} \), \( u \) is a concave transform of \( \hat{u} \), and then that \( (1) \) requires \( b > \hat{b} = 0 \).

Part 1.

We consider acts \( f \) with the property that there exists a neighborhood \([m(f) - 2\delta, m(f) + 2\delta]\) of \( m(f) \) that is not in the support of \( f \). Call this Property 1.
For any \( f \) satisfying Property 1, we can partition \( S \) into disjoint events \( E \) (elation) and \( E' \) (the complement of \( E \)) such that

\[
f(s) < m(f) - 2\delta \quad s \in E'
\]

\[
f(s) > m(f) + 2\delta \quad s \in E
\]

Hence, for any \( g, |g(s)| < \delta \), we have

\[
f(s) + g(s) < m(f) - \delta \quad s \in E'
\]

\[
f(s) + g(s) > m(f) + \delta \quad s \in E
\]

and hence

\[
u(m(f + g)) - u(m(f)) = \int_{E'} u((f + g)(s)) - u(f(s)) \, d\pi + \int_E (1 - b) (u((f + g)(s)) - u(f(s))) \, d\pi
\]

Now by Unboundedness, we can choose \( f \) satisfying Property 1 for both \( \preceq \) and \( \preceq \) and such that \( \pi(E) \) is arbitrarily small for both \( \preceq \) and \( \preceq \). Since the integrand in the second term on the RHS is bounded, this term can be made arbitrarily close to zero, in particular smaller in absolute value than any \( \Delta > 0 \).

Suppose for some \( g, |g(s)| < \delta \), we have \( m(f) = m(f + g) \). Then

\[
\int_{E'} u((f + g)(s)) - u(f(s)) \, d\pi + \int_E (1 - b) (u((f + g)(s)) - u(f(s))) \, d\pi = 0
\]

implies

\[
\left| \int_S u((f + g)(s)) - u(f(s)) \, d\pi \right| < \Delta
\]

We can now apply the argument of the SEU proposition to show that \( \preceq \) is everywhere at least as uncertainty averse as \( \preceq \) only if \( \pi(A) = \hat{\pi}(A) \) for all \( A \in \mathcal{E} \) and \( u \) is a concave transform of \( \hat{u} \)

**Part 2.**

It is trivial that we require \( b \geq \hat{b} \). Suppose \( b \geq \hat{b} > 0 \), and that \( \pi(A) = \hat{\pi}(A) \) for all \( A \in \mathcal{E}, u \) is a concave transform of \( \hat{u} \). Then the Gul result shows that for any non-trivial \( f \), \( m(f) < \hat{m}(f) \). Let \( E, \hat{E} \) be the elation events. By choosing \( f \) with a probability mass in the
interval \([m(f), \hat{m}(f)]\), we can make the measure of \(E - \hat{E}\) as large as we like relative to \(E'\) and \(\hat{E}\). Choose \(\delta\) and \(\delta'\) to define the event \(g\)
\[
g(s) = \begin{cases} 
\delta & s \in \hat{E} \\
-\delta' & s \in \hat{E}'
\end{cases}
\]
such that
\[
(1 - b) \left( \int_{\hat{E}} (u(f + \delta) - u(f)) \, d\pi \right) - \int_{E'} (u(f) - u(f - \delta')) \, d\pi = 0
\]
so that \(\hat{m}(f) = \hat{m}(f + g)\). Note that this can be rewritten as
\[
(1 - b) \left( \int_{\hat{E}} (u(f + \delta) - u(f)) \, d\pi \right) - \int_{E - \hat{E}} (u(f) - u(f - \delta')) \, d\pi \\
- \int_{E'} (u(f) - u(f - \delta')) \, d\pi = 0
\]
Hence provided the measure of \(E - \hat{E}\) is large enough
\[
(1 - b) \left( \int_{\hat{E}} (u(f + \delta) - u(f)) \, d\pi \right) - (1 - b) \left( \int_{E - \hat{E}} (u(f) - u(f - \delta')) \, d\pi \right) \\
- \int_{E'} (u(f) - u(f - \delta')) \, d\pi > 0
\]
so that \(m(f + g) > m(f)\) and hence \(\succsim\) is not everywhere at least as uncertainty averse as \(\succsim\). 

\[\blacksquare\]

**Proof of Proposition 9.**

Consider any pair of comonotonic simple acts \(f\) and \(g\), for which \(g - f = \alpha A (-\beta)\), for some \(A\) in \(\mathcal{E}\), and for some \(\alpha, \beta > 0\). Further suppose \(g \sim f\). We need to show that \(g \succsim f\). Applying (1), \(f \succsim g\) implies
\[
\int_{-\infty}^{\infty} [\nu (\{ s : u(f(s)) \geq w \}) - \nu (\{ s : u(g(s)) \geq w \})] \, dw \geq 0
\]
\[
0 \leq \int_{u(\mathbb{E})}^{u(\mathbb{S} - A)} [\nu (\{ s : u(f(s)) \geq w \}) - \nu (\{ s : u(f(s) - \beta) \geq w \})] \, dw \\
+ \int_{u(\mathbb{S}^{A})}^{u(\mathbb{S})} [\nu (\{ s : u(f(s)) \geq w \}) - \nu (\{ s : u(f(s) + \alpha) \geq w \})] \, dw \\
\leq \beta u'(\mathbb{S} - A) + \alpha u'(\mathbb{S}^{A})
\]

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where \( \underline{x} = \inf_{s \in S} f(s) \), \( \overline{x}_{S-A} = \sup_{s \in A} f(s) \), \( \underline{x}^A = \inf_{s \in A} f(s) \), \( \overline{x} = \sup_{s \in S} f(s) \). And so
\[
\frac{\nu(A) \alpha u'(\underline{x}^A)}{(1 - \nu(A)) \beta u'(\overline{x}_{S-A})} \geq 1.
\]

The preference between \( g \) and \( f \) for \( \preceq \) is determined by the sign of
\[
\int_{-\infty}^{\infty} \left[ \hat{\nu} \left( \{ s : \hat{u}(f(s)) \geq w \} \right) - \hat{\nu} \left( \{ s : \hat{u}(g(s)) \geq w \} \right) \right] dw
\]
\[
\tag{10}
\]
Now by (2)
\[
\frac{\hat{\nu}(A) \hat{u}'(\overline{x})}{(1 - \hat{\nu}(A)) \hat{u}'(\underline{x})} \geq \frac{\nu(A) u'(\underline{x}^A)}{(1 - \nu(A)) u'(\overline{x}_{S-A})},
\]

hence
\[
- (1 - \hat{\nu}(A)) \beta \hat{u}'(\underline{x}) + \hat{\nu}(A) \alpha \hat{u}'(\overline{x}) \geq 1
\]
\[
\tag{11}
\]
But by the concavity of \( \hat{u} \), (10) is greater than or equal to righthand side of inequality (11), which in turn implies that (10) is greater than or equal to zero and thus \( f \preceq g \), as required.

**Proof of Proposition 13** Fix two comonotonic acts \( f \) and \( g \).

We first consider the case in which the two acts exhibit a “finite-crossing” property in the sense that there exists a finite partition, \( \{ E^1, \ldots, E^K \} \) that is ordered with respect to \( f \) (and hence also with respect to \( g \)) and such that, for each \( k \), either
\[
(a) \quad g(s) \geq f(s) \quad \forall s \in E^k
\]
or
\[
(b) \quad g(s) \leq f(s) \quad \forall s \in E^k
\]
If (a) holds on \( E^K \) and (b) holds on \( E^1 \), then the standard analysis under risk applies - note that by selecting the probability distribution over states appropriately, we can always ensure that \( g \) is more risky than \( f \) in the sense of Rothschild-Stiglitz. That is, there exists a probability distribution \( \mu(\cdot) \) defined over \( \mathcal{E} \) such that for all \( x \)
\[
\int_{0}^{x} (\mu(\{ s : f(s) \leq y \}) - \mu(\{ s : g(s) \leq y \})) dy \leq 0
\]
and
\[
\int_{0}^{\infty} (\mu(\{ s : f(s) \leq y \}) - \mu(\{ s : g(s) \leq y \})) dy = 0
\]
Hence, as Machina and Pratt (1997) show there exists a sequence of simple mean preserving spreads
Consider the case where (a) holds on $E^1$ and $E^K$. (note that this includes the case when $g(s) \geq f(s) \ \forall s$). Now consider any descending series of non-empty sets $E^1 = A^1 \supset A^2 \ldots$. such that

(i)  
$$\bigcap_{i=1}^{\infty} A^i = \emptyset$$

(ii)  
$$\sup \{ f(s) : s \in A^i \} \leq \inf \{ f(s) : s \in E^1 \setminus A^i \}$$

Small-event continuity ensures that such a series exists. Now consider acts $h^i$ such that for some $\delta > 0$.

$$h^i(s) = \begin{cases} g(s) & s \in S \setminus A^i \\ f(s) + \delta & s \in A^i \end{cases}$$

Then, by the argument already given, $h^i$ is in the transitive closure of the elementary transfer relation, and $h^i \to g$.  

\textbf{Proof of Proposition 16} (1) Consider any fixed $d$, and $c < \tilde{c}^\star$. We have 

$$f(p, d, c) = \hat{f}^\star(p, d, c) \overline{\cal U} \hat{f}^\star(p, d, \tilde{c}^\star)$$

where the equality follows from comparability and the risk ordering from 7. Now, since 

$$\hat{f}^\star(p, d, \tilde{c}^\star) \succsim \hat{f}^\star(p, d, c)$$

by the optimality of $\tilde{c}^\star$ and hence 

$$f^\star(p, d, \tilde{c}^\star) \succsim f^\star(p, d, c)$$

by comparative uncertainty aversion. Hence, $c$ cannot be strictly optimal for $\succsim$ and since this is true for all $c < \tilde{c}^\star$, we must have $c^\star \geq \tilde{c}^\star$ as desired.

(2) Follows by integrating $\frac{\partial\text{WTP}(dp)}{\partial d}$ from $d^0$ to $d^1$ and applying 8 and 9.  

\textbf{Proof of Proposition 17}: Consider an increase from $p^0$ to $p^1$. Observe by constant absolute uncertainty aversion that, for all $c$

$$q^\star\left(p^1, d, \frac{cp^1}{p^0}\right) = q^\star(p^0, d, c)$$

Now consider $c \geq C(q^\star(p^0, d), d, p^0)$ By optimality

$$\eta(q^\star(p^0, d)) - C(q^\star(p^0, d), d, p^0) \succsim \eta(q^\star(p^0, d, c)) - c$$
Now since
\[
(c - C(q^*(p^0, d), d, p^0)) \left(\frac{p^1}{p^0} - 1\right) \geq 0
\]
Constant absolute uncertainty aversion implies
\[
\eta(q^*(p^0, d)) - \frac{p^1}{p^0} C(q^*(p^0, d), d, p^0) \geq \eta(q^*(p^0, d, e)) - \frac{cp^1}{p^0}
\]
So that expenditure level \( \frac{cp^1}{p^0} \) cannot be optimal. Hence, the new optimal expenditure level must be less than \( \frac{p^1}{p^0} C(q^*(p^0, d), d, p^0) \) implying a reduction in the optimal \( q \). ■

References


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