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Economic Theory

Exposita Notes

Jointly radial and translation homothetic preferences: generalized constant risk aversion

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Summary. The paper identifies the structural restrictions on preferences required for them to exhibit both translation homotheticity in particular direction and radial homotheticity. The results are illustrated by an application to an asset allocation problem in the absence of riskless asset.

Keywords and Phrases: Translation homotheticity, Radial homotheticity.

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1 Introduction

Following the early work of Blackorby and Donaldson (1980), Chambers and Färe (1998) have introduced and characterized the notions of translation homothetic preference and production structures. Among other results they have shown that constant absolute risk aversion corresponds to translation homotheticity in the direction of the riskless prospect.

The present paper extends the analysis of Chambers and Färe (1998) to determine necessary and sufficient conditions for a production or preference structure to exhibit both translation homotheticity and radial (Shephard) homotheticity. For the sake of concreteness, we work in terms of the Quiggin and Chambers (1998)

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extension of the Yaari (1969) representation of preferences over state-contingent incomes, and thus we speak of a notion of generalized constant risk aversion. The resulting analysis, however, is analogous to that for the consumer problem or the producer problem, with risk-neutral probabilities, familiar from finance theory, playing a role analogous to that of relative prices. The main differences in interpretation arise from the central role played in the analysis of choice under uncertainty by concepts of risk-aversion, normally interpreted as a preference for receipt of a given nonstochastic income over some state-contingent income having the same expected value, calculated with respect to some given set of subjective or objective probabilities. While we work in terms of preferences over stochastic outcomes, it is a simple matter to extend our results to either non-stochastic production or preference structures.

To illustrate, the possible importance of preferences that are jointly translation homothetic and radial homothetic, we briefly consider applications of this type of preference structure to portfolio choice for the case where no riskless asset is available.

2 Notation

We consider preferences over state-contingent income distributions $\mathbf{y} \in \Re^{\Omega}$. Our focus is on the case where Ω is a finite set $\{1, ..., S\}$, and the space of random variables is thus $\Re^S$. The unit vector is denoted $\mathbf{1} = (1, 1, ..., 1)$, and $\mathcal{P} \subset \Re^S_+$ denotes the probability simplex.

Preferences over state-contingent incomes are given by an ordinal mapping $W : Y^S \to \Re$. W is continuous, nondecreasing, and quasi-concave in $\mathbf{y}$. Quasi-concavity ensures that the least-as-good sets of the preference mapping

$$V(w) = \{\mathbf{y} : W(\mathbf{y}) \geq w\}$$

are convex, and the fact that W is nondecreasing ensures that $V(w) + \Re^S_+ = V(w)$. The *benefit function*, $B : \Re \times \Re^S \to \Re$, is defined for direction $\mathbf{g} \in \Re^S_+$ by:

$$\begin{aligned} B(w, \mathbf{y}; \mathbf{g}) &= \bar{B}(V(w); \mathbf{y}; \mathbf{g}) \\ &= \max\{\beta \in \Re : \mathbf{y} - \beta \mathbf{g} \in V(w)\} \\ &= \max\{\beta \in \Re : W(\mathbf{y} - \beta \mathbf{g}) \geq w\} \end{aligned}$$

if $\mathbf{y} - \beta \mathbf{g} \in V(w)$ for some β and $-\infty$ otherwise (Blackorby and Donaldson, 1980; Luenberger, 1992).[1] The properties of $B(w, \mathbf{y}; \mathbf{g})$ are well known (Luenberger, 1992; Chambers, Chung, and Färe, 1996) and are summarized for later use in the following lemma:

Lemma 1. $B(w, \mathbf{y}; \mathbf{g})$ *satisfies:*

a) $B(w, \mathbf{y}; \mathbf{g})$ *is nonincreasing in* w *and nondecreasing and concave in* $\mathbf{y}$*;*

[1] When $\mathbf{g} = \mathbf{1}$ the benefit function corresponds to the *translation function* introduced by Blackorby and Donaldson (1980).

b) $B(w, \mathbf{y} + \alpha\mathbf{g}; \mathbf{g}) = B(w, \mathbf{y}; \mathbf{g}) + \alpha$, $\alpha \in \Re$ *(the translation property);*
c) $B(w, \mathbf{y}; \mathbf{g}) \geq 0 \Leftrightarrow \mathbf{y} \in V(w)$*;*
d) $B(w, \mathbf{y}; \mathbf{g})$ *is jointly continuous in* $\mathbf{y}$ *and* w *in the interior of the region* $\Re \times Y^S$ *where* $B(w, \mathbf{y}; \mathbf{g})$ *is finite;*
e) $B(w, \mathbf{y}; \mu\mathbf{g}) = \mu^{-1} B(w, \mathbf{y}; \mathbf{g}), \quad \mu > 0.$

Proof.

a) That B is nonincreasing in w and nondecreasing in $\mathbf{y}$ follows from W nondecreasing in $\mathbf{y}$. To establish concavity, by the definition of B

$$\mathbf{y}^0 - B(w, \mathbf{y}^0; \mathbf{g})\mathbf{g} \in V(w),$$
$$\mathbf{y}^1 - B(w, \mathbf{y}^1; \mathbf{g})\mathbf{g} \in V(w),$$

which by the convexity of V implies for $\lambda \in (0, 1)$

$$\lambda\mathbf{y}^0 + (1 - \lambda)\,\mathbf{y}^1 - \left[\lambda B(w, \mathbf{y}^1; \mathbf{g}) + (1 - \lambda)\, B(w, \mathbf{y}^0; \mathbf{g})\right] \mathbf{g} \in V(w),$$

and concavity follows immediately from the definition of B.

b)

$$\begin{aligned} B(w, \mathbf{y} + \alpha\mathbf{g}; \mathbf{g}) &= \max\{\beta \in \Re : \mathbf{y} + \alpha\mathbf{g} - \beta\mathbf{g} \in V(w)\} \\ &= \max\{\beta \in \Re : \mathbf{y} - (\beta - \alpha)\,\mathbf{g} \in V(w)\} \\ &= \max\{\beta - \alpha + \alpha \in \Re : \mathbf{y} - (\beta - \alpha)\,\mathbf{g} \in V(w)\} \\ &= \alpha + \max\{\beta - \alpha \in \Re : \mathbf{y} - (\beta - \alpha)\,\mathbf{g} \in V(w)\}. \end{aligned}$$

c) $\Leftarrow$ follows from supposing $\mathbf{y} \in V(w)$ and the definition of B. $\Rightarrow$ follows from the fact that W nondecreasing implies $W(\mathbf{y}) \geq W(\mathbf{y} - B(w, \mathbf{y}; \mathbf{g})\,\mathbf{g}) \geq w$. *d)* follows from the theorem of the maximum (Berge, 1963) and the continuity of W. e) For $\mu > 0$

$$\begin{aligned} B(w, \mathbf{y}; \mu\mathbf{g}) &= \max\{\beta \in \Re : \mathbf{y} - \beta\mu\mathbf{g} \in V(w)\} \\ &= \max\{\beta\frac{\mu}{\mu} \in \Re : \mathbf{y} - \beta\mu\mathbf{g} \in V(w)\} \\ &= \mu^{-1} \max\{\beta\mu \in \Re : \mathbf{y} - \beta\mu\mathbf{g} \in V(w)\}. \end{aligned}$$ □

The benefit function affords a general method for obtaining alternative representations of preferences. For example, the certainty equivalent is the particular case:

$$\begin{aligned} e(\mathbf{y}) &= \min\{c > 0 : W(c\mathbf{1}) \geq W(\mathbf{y})\} \\ &= -B(W(\mathbf{y}), \mathbf{0}; \mathbf{1}) \end{aligned}$$

The certainty equivalent trivially satisfies Aczél's (1990) *agreement property*

$$e(\mu\mathbf{1}) = \mu, \quad \mu \in \Re.$$

The use of certainty equivalents (generalized mean values) as representations of preferences has been discussed by Chew (1982).

We define the *expected-value function* $E : \mathcal{P} \times \Re \to \Re$. It is defined by

$$\begin{aligned} E(\pi,w) &= \inf_{\mathbf{y}} \{\pi\mathbf{y} : \mathbf{y} \in V(w)\} \quad \pi \in \mathcal{P} \\ &= \inf_{\mathbf{y}} \{\pi\mathbf{y} : B(w,\mathbf{y};\mathbf{g}) \geq 0\} \quad \pi \in \mathcal{P} \\ &= \inf_{\mathbf{y}} \{\pi\mathbf{y} - B(w,\mathbf{y};\mathbf{g})\,\pi\mathbf{g}\} \quad \pi \in \mathcal{P}. \end{aligned}$$

if there exists some $\mathbf{y}$ such that $B(w,\mathbf{y};\mathbf{g}) \geq 0$ and ∞ otherwise (Chambers and Quiggin, 2001). $E(\pi,w)$ is concave and nondecreasing on $\mathcal{P}$ and continuous on the relative interior of the region of $\mathcal{P}$ where it is finite, and continuous and nondecreasing in w in the interior of region where it is finite. From Luenberger (1992) and Chambers, Chung and Färe (1996) we also have the corresponding dual relation between the expected-value function and the benefit function

$$B(w,\mathbf{y};\mathbf{g}) = \inf_{\pi \in \mathcal{P}} \left\{ \frac{\pi\mathbf{y} - E(\pi,w)}{\pi\mathbf{g}} \right\} \tag{1}$$

$$E(\pi,w) = \inf_{\mathbf{y}} \{\pi\mathbf{y} - B(w,\mathbf{y};\mathbf{g})\,\pi\mathbf{g}\} \quad \pi \in \mathcal{P}. \tag{2}$$

Moreover, we also have the following dual relationship between the expected-value function and the least-as-good set

$$V(w) = \cap_{\pi \in \mathcal{P}} \{\mathbf{y} : E(\pi,w) \leq \pi\mathbf{y}\} \tag{3}$$

For a proper[2] concave function $f : \Re^S \to \Re$, its *superdifferential* at $\mathbf{x}$ is the closed, convex set:

$$\partial f(\mathbf{x}) = \left\{\mathbf{v} \in \Re^S : f(\mathbf{x}) + \mathbf{v}(\mathbf{z} - \mathbf{x}) \geq f(\mathbf{z}) \text{ for all } \mathbf{z}\right\}. \tag{4}$$

The elements of $\partial f(\mathbf{x})$ are referred to as *supergradients*.

By the duality between the benefit function and the expected-value function (Rockafellar, 1970)

$$\frac{\pi}{\pi\mathbf{g}} \in \partial B(e,\mathbf{y};\mathbf{g}) \Longleftrightarrow \mathbf{y} \in \partial E(\pi,e) \tag{5}$$

in the relative interior of their domains.

The translation property of the benefit function (Lemma 1b) allows us to establish an analogue of Euler's theorem for functions translatable in the direction of $\mathbf{g}$:

Lemma 2. *Let*

$$\mathbf{p}(w,\mathbf{y},\mathbf{g}) \in \partial B(w,\mathbf{y},\mathbf{g}).$$

[2] A concave function f is proper if $f(\mathbf{x}) < \infty$ for every $\mathbf{x}$ and if there exists an $\mathbf{x}$ such that $f(\mathbf{x}) > -\infty$.(Rockafellar, 1970).

Then

$$\sum_{s\in\Omega} p_s(w,\mathbf{y},\mathbf{g})\, g_s = 1.$$

Moreover,

$$\mathbf{p}(w,\mathbf{y}+\delta\mathbf{g},\mathbf{g}) = \mathbf{p}(w,\mathbf{y};\mathbf{g}) \quad \delta\in\Re.$$

Proof. By the translation property (Lemma 1b):

$$B(w,\mathbf{y}+\delta\mathbf{g},\mathbf{g}) = B(w,\mathbf{y},\mathbf{g})+\delta.$$

Hence if $\mathbf{v}\in\partial B(w,\mathbf{y},\mathbf{g})$ and $\mathbf{z}=\mathbf{y}+\delta\mathbf{g}$, $\mathbf{z}^{*}=\mathbf{y}-\delta\mathbf{g}$, the definition of the superdifferential implies

$$\begin{aligned} B(w,\mathbf{y},\mathbf{g}) + \mathbf{v}(\mathbf{z}-\mathbf{y}) &\geq B(w,\mathbf{z},\mathbf{g}) \\ B(w,\mathbf{y},\mathbf{g}) + \mathbf{v}(\mathbf{z}^{*}-\mathbf{y}) &\geq B(w,\mathbf{z}^{*},\mathbf{g}) \end{aligned}$$

or

$$\begin{aligned} B(w,\mathbf{y},\mathbf{g}) + \delta\mathbf{v}\mathbf{g} &\geq B(w,\mathbf{y},\mathbf{g}) + \delta \\ B(w,\mathbf{y},\mathbf{g}) - \delta\mathbf{v}\mathbf{g} &\geq B(w,\mathbf{y},\mathbf{g}) - \delta \end{aligned}$$

so that

$$\sum_{s\in\Omega} v_s g_s = 1.$$

For the second part,

$$\begin{aligned} \partial B(w,\mathbf{y}+\delta\mathbf{g},\mathbf{g}) &= \left\{ \begin{array}{c} \mathbf{v}: B(w,\mathbf{y}+\delta\mathbf{g},\mathbf{g}) + \mathbf{v}(\mathbf{z}+\delta\mathbf{g}-[\mathbf{y}+\delta\mathbf{g}]) \\ \geq B(w,\mathbf{z}+\delta\mathbf{g},\mathbf{g}) \text{ for all } \mathbf{z}+\delta\mathbf{g} \end{array} \right\} \\ &= \{\mathbf{v}: B(w,\mathbf{y},\mathbf{g}) + \mathbf{v}(\mathbf{z}-\mathbf{y}) \geq B(w,\mathbf{z},\mathbf{g}) \text{ for all } \mathbf{z}\} \\ &= \partial B(w,\mathbf{y},\mathbf{g}), \end{aligned}$$

where the second equality follows by the translation property.

Define $\pi(\mathbf{y};\mathbf{g}) = \mathbf{p}(W(\mathbf{y}),\mathbf{y};\mathbf{g})$. Any element of $\pi(\mathbf{y};\mathbf{g})$ can be interpreted as a vector supporting contingent state-claim prices, which in turn are closely related to the concept of risk-neutral or shadow probabilities (Peleg and Yaari, 1975; Nau, 2001; Chambers and Quiggin, 2001). Expression (5) demonstrates that for any

$$\pi\in\pi(\mathbf{y};\mathbf{g}),$$

the ratios $\frac{\pi_s}{\pi_k}$ can be interpreted as relative risk neutral probabilities. However, Lemma 2 establishes that $\pi\in\pi(\mathbf{y};\mathbf{g})$ cannot be interpreted as a vector of risk-neutral probabilities because for $\pi\in\pi(\mathbf{y};\mathbf{g})$

$$\sum_{s} \pi_s g_s = 1,$$

and hence $\pi(\mathbf{y};\mathbf{g})$ is only guaranteed to lie in the unit simplex when $\mathbf{g}=\mathbf{1}$.

3 Translation homothetic, CRRA preferences

One of the more enduring concepts in the analysis of preferences under uncertainty is that of constant absolute risk aversion which asserts that

$$W(\mathbf{y}) = W(\mathbf{y}') \Leftrightarrow W(\mathbf{y} + \delta\mathbf{1}) = W(\mathbf{y}' + \delta\mathbf{1}),$$

or equivalently in terms of least-as-good sets for ordinal preferences (Chambers and Färe, 1998)

$$V(w) = w\mathbf{1} + V^0,$$

where V^0 is a least-as-good set which is independent of w. Intuitively, this means that an individual's fundamental attitudes towards risk are not altered by receiving any additional units of the traditionally safe asset. More generally, one can think of preference structures where the fundamental attitudes towards risk are unchanged by receiving additional units of a fixed, but arbitrary risky asset, or in other words

$$V(w) = w\mathbf{g} + V^0.$$

Chambers and Färe (1998) define such preferences to be *translation homothetic in the direction of* $\mathbf{g}$.

The following lemma is due to Chambers and Färe (1998):

Lemma 3. *Ordinal preferences are translation homothetic in the direction of* $\mathbf{g}$ *if and only if*

$$B(w, \mathbf{y}; \mathbf{g}) = B^0(\mathbf{y}; \mathbf{g}) - w,$$
$$E(\pi, w) = w\pi\mathbf{g} + E^0(\pi),$$

where B^0 *and* E^0 *are the benefit and expected value functions associated with* V^0.

Remark 4. By Lemma 1 and Lemma 3, it follows that when preferences are translation homothetic in the direction of $\mathbf{g}$,

$$W(\mathbf{y}) = B^0(\mathbf{y}; \mathbf{g}),$$

from which follows the well-known consequence that the preference functional itself is translation homothetic in the direction of $\mathbf{g}$ by the translation property of benefit functions.

An immediate consequence of Lemma 3 is the fact that the supporting state-claim prices are invariant to additions or subtraction of units of the risky asset $\mathbf{g}$. In visual terms, this implies that indifference curves are all parallel in the direction of $\mathbf{g}$. Several well-known translation homothetic preferences illustrate. Consider first the class of Leontief preferences

$$W(\mathbf{y}) = \min\left\{\frac{y_1}{g_1}, \ldots, \frac{y_S}{g_S}\right\}.$$

These preferences are obviously translation homothetic in the direction of $\mathbf{g}$ because

$$\begin{aligned} W(\mathbf{y}+\delta\mathbf{g}) &= \min\left\{\frac{y_1+\delta g_1}{g_1},...,\frac{y_S+\delta g_s}{g_S}\right\} \\ &= \min\left\{\frac{y_1}{g_1},...,\frac{y_S}{g_S}\right\}+\delta. \end{aligned}$$

Next consider the class of *risk-neutral preferences,* expressed here in g-normalized form

$$W(\mathbf{y}) = \frac{\sum_s \pi_s y_s}{\sum_s \pi_s g_s},$$

which are also trivially translation homothetic. The superdifferential for the Leontief preference function at all points on the expansion path

$$\frac{y_1}{g_1} = ... = \frac{y_S}{g_S}$$

consists of $\mathcal{P}$, while the superdifferential for the $\mathbf{g}-$normalized risk-neutral preferences are simply its gradient

$$\frac{\pi}{\pi\mathbf{g}}.$$

The invariance of the superdifferentials to translations in the direction of $\mathbf{g}$ is a characteristic of translation homothetic preferences. Our next result characterizes the class of supporting state-claim prices for preferences that are translation homothetic.

Theorem 5. *Ordinal preferences are translation homothetic in the direction of* $\mathbf{g}$ *if and only if*

$$\pi(\mathbf{y}+\beta\mathbf{g};\mathbf{g}) = \pi(\mathbf{y};\mathbf{g}) \quad \beta\in\Re.$$

Proof. By translation homotheticity

$$\begin{aligned} \pi(\mathbf{y};\mathbf{g}) &= \mathbf{p}(W(\mathbf{y}),\mathbf{y},\mathbf{g}) \in \partial B(W(\mathbf{y}),\mathbf{y},\mathbf{g}) \\ &= \partial B^0(\mathbf{y};\mathbf{g}). \end{aligned}$$

Now apply the second part of Lemma 2. □

By results in Chambers and Färe (1998), Quiggin and Chambers (1998), and Chambers and Quiggin (2001), the usual notion of constant relative risk aversion is equivalent to homotheticity of state-contingent preferences and hence to linear homogeneity of the certainty equivalent. We state this fact in the following lemma

Lemma 6. *Ordinal preferences exhibit constant relative risk aversion if and only if*

$$V(w) = wV^1,$$
$$E(\pi,w) = wE^1(\pi)$$
$$B(w,\mathbf{y};\mathbf{g}) = wB^1\left(\frac{\mathbf{y}}{w};\mathbf{g}\right),$$

where $w > 0$, $V^1 \subset \Re_+^S$ is a least-as-good set which is independent of w, and E^1 and B^1 are the expected-value and benefit functions associated with V^1.

Safra and Segal (1998) define preferences that simultaneously satisfy constant relative risk aversion and constant absolute risk aversion as exhibiting constant risk aversion. Generalizing Safra and Segal (1998), we refer to preferences satisfying constant relative risk aversion and translation homotheticity in the direction of $\mathbf{g}$ as $\mathbf{g}-$*generalized constant risk averse.* On the basis of Lemma 6 and Lemma 3, we can now establish the central result of this part of the paper which generalizes the result for constant risk aversion established in Chambers and Quiggin (2001):

Theorem 7. *If $V(w)$ is nonempty, ordinal preferences are consistent with $\mathbf{g}-$generalized constant risk aversion if and only if*

$$E(\pi,e) = \begin{cases} e & \pi \in \mathcal{P}* \subseteq \mathcal{P} \\ -\infty & \pi \notin \mathcal{P}* \end{cases},$$
$$B(w,\mathbf{y};\mathbf{g}) = \inf_{\pi}\left\{\frac{\pi}{\pi\mathbf{g}}\mathbf{y} : \pi \in \mathcal{P}^*\right\} - w$$
$$W(\mathbf{y}) = \inf_{\pi}\left\{\frac{\pi}{\pi\mathbf{g}}\mathbf{y} : \pi \in \mathcal{P}^*\right\}$$

Proof. If preferences are consistent with constant relative risk aversion

$$E(\pi,\mu w) = \mu E(\pi,w) \quad \mu > 0.$$

By translation homotheticity in the direction of $\mathbf{g}$, this requires

$$\mu E^0(\pi) = E^0(\pi).$$

There are three possibilities, $E^0(\pi) = -\infty$, $E^0(\pi) = 0$, and $E^0(\pi) = \infty$. The last implies that there is no $\mathbf{y}$ such that $\mathbf{y} \in V^1$, and so V^1 must always be empty and the expected-value function must always be infinity. Hence, we conclude that if $V(w)$ is not empty

$$E(\pi,e) = \begin{cases} e & \pi \in \mathcal{P}* \subseteq \mathcal{P} \\ -\infty & \pi \notin \mathcal{P}* \end{cases}$$

which establishes the first expression. The second expression follows by applying (1), and the third by the Remark. This establishes necessity. Sufficiency is obvious. □

4 Application to asset demands

4.1 Monotonic spreads

We begin with a new definition:

Definition 8. *Let* $\mathbf{y} \in \Re_+^S$. *Then* $\mathbf{y}$ *is monotonic with respect to* $\mathbf{g}$ *if for all* $s, t \in \{1...S\}$

$$(y_s - y_t)(g_s - g_t) \geq 0$$

Note that complementary slackness is not required, so any $\mathbf{y} \in \Re^S$ is monotonic with respect to $\mathbf{1}$. Let,

$$M(\mathbf{g}) = \left\{\mathbf{y} \in \Re_+^S : \mathbf{y} \text{ is monotonic with respect to } \mathbf{g}\right\}.$$

We observe that

$$\begin{aligned} M(t\mathbf{g}) &= M(\mathbf{g}) \quad t > 0 \\ M(\mathbf{g}+\delta\mathbf{1}) &= M(\mathbf{g}) \\ M(\mathbf{g}+\mathbf{m}) &\subseteq M(\mathbf{g}) \quad \mathbf{m} \in M(\mathbf{g}). \end{aligned}$$

The first two results are trivial. For the third, consider $\mathbf{y} \in M(\mathbf{g}+\mathbf{m})$ and any $s, t \in \{1...S\}$. The monotonicity condition for $\mathbf{g}$ holds trivially if $g_s = g_t$ so suppose wlog that $g_s > g_t$. Then $m_s \geq m_t$ so $m_s + g_s > m_t + g_t$ and hence $y_s \geq y_t$ as required. To see that the inclusion may be strict consider $\mathbf{g} = \mathbf{1}$ and $\mathbf{m}$ not on the bisector.

4.2 The portfolio problem

We now consider applications to the problem of portfolio choice. For simplicity we consider the simplest class of problems, in which a fixed sum of wealth m must be allocated between two assets, with no short-selling. One of the assets yields returns $\mathbf{g}$. The other has returns $\mathbf{y} \in M(\mathbf{g})$. Since the two return distributions are comonotonic, there is no idiosyncratic risk and no gain from diversification. Let α_1 be the holding of the asset defined by $\mathbf{g}$ and α_2 the holding of the other asset and let p_1, p_2 be the asset prices. Thus, the problem is

$$max_{(\alpha_1,\alpha_2)\in\Re_+} \{e(\alpha_1\mathbf{g}+\alpha_2\mathbf{y}) : p_1\alpha_1+p_2\alpha_2 \leq m\}$$

The choice set may be written as $A \subseteq \Re_+^2$

$$A(p_1, p_2, m) = \{(\alpha_1, \alpha_2) : p_1\alpha_1+p_2\alpha_2 \leq m\}$$

and the optimal solution may be written as $\alpha^*(p_1, p_2, m)$. We focus on the case when preferences are translation homothetic in the direction of $\mathbf{g}$. Now if $\alpha \in A(p_1, p_2, m)$,

$$e(\alpha_1\mathbf{g}+\alpha_2\mathbf{y}) \leq e(\alpha_1^*\mathbf{g}+\alpha_2^*\mathbf{y})$$

and therefore, by translation homotheticity, for any δ

$$e\left((\alpha_1+\delta)\,\mathbf{g}+\alpha_2\mathbf{y}\right) \leq e\left((\alpha_1^*+\delta)\,\mathbf{g}+\alpha_2^*\mathbf{y}\right)$$

Focusing on the case $\delta > 0$, this implies that if $\alpha^*(p_1, p_2, m)$ is a unique interior solution, and $m' = m + \delta p_1$

$$\alpha^*(p_1, p_2, m') = (\alpha_1^*(p_1, p_2, m) + \delta, \alpha_2^*(p_1, p_2, m))$$

That is, for an interior solution, changes in m have no impact on the optimal choice of α_2.

Next, consider a compensated change in p_1. That is, given an initial (p_1, p_2, m), choose (p_1', p_2, m') such that

$$p_1'\alpha_1^*(p_1, p_2, m) + p_2\alpha_2^*(p_1, p_2, m) = m'$$

The first-order conditions for a unique interior optimum imply

$$\frac{e_{\mathbf{g}}}{e_{\mathbf{y}}} = \frac{p_1}{p_2}$$

so, assuming that the second-order conditions are satisfied, a compensated reduction in p_1 must imply an increase in α_1 and a reduction in α_2. Since, when preferences are translation homothetic in the direction of $\mathbf{g}$, the optimal choice of α_2 is independent of m, we have:

Proposition 9. *Assume that preferences are translation homothetic in the direction of* $\mathbf{g}$. *For an interior solution, an increase in* p_1 *leads to a reduction in* α_1 *and an increase in* α_2

The same logic may be used to show that, for an interior solution, an increase in p_2 leads to a reduction in α_2, but in the absence of compensation, the impact on α_1 is ambiguous.

For the case of constant risk aversion, Chambers and Quiggin (2001) generalize the results of Yaari (1987) showing that the optimal solution involves 'plunging'. That is, either $\alpha_1^* = 0$ or $\alpha_2^* = 0$. We now observe:

Corollary 10. *If preferences satisfy* $\mathbf{g}-$ *generalized constant risk aversion, the optimal solution*

$$max_{(\alpha_1,\alpha_2)\in\Re_+}\left\{e\left(\alpha_1\mathbf{g}+\alpha_2\mathbf{y}\right):p_1\alpha_1+p_2\alpha_2 \leq m\right\}$$

has either $\alpha_1^* = 0$ *or* $\alpha_2^* = 0$.

5 Conclusion

This paper has characterized preference and production structures that are jointly radial homothetic and translation homothetic. The usefulness of this characterization has been illustrated by an application of such preferences to the portfolio problem.

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