TWO-PARAMETER DECISION MODELS AND RANK-DEPENDENT EXPECTED UTILITY

Michael Ormiston

ARIZONA STATE UNIVERSITY

John Quiggin

AUSTRALIAN NATIONAL UNIVERSITY

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• Introduction

Since the classic work of John von Neumann and Oskar Morgenstern (1944), the Expected Utility (EU) theory of choice under uncertainty has had as its main competitor the mean-standard deviation (MS)[1] model proposed by Markowitz. The criticisms of Karl Borch (1969) and Martin Feldstein (1969), along with technical innovations in EU theory such as the risk premium analysis of Kenneth Arrow () and John Pratt (1964) and the stochastic dominance analysis of Michael Rothschild and Joseph Stiglitz (1970, 1971) eliminated mean-variance as a theoretical contender during the 1970's.

Jack Meyer (1987) presented a partial rehabilitation of the mean-variance approach. In essence, Meyer observed that most comparative static analysis, such as the analysis of the theory of the firm by Agnar Sandmo (1971), dealt with choices over sets cumulative distribution functions which differed only by location and scale (LS) parameters. For example Sandmo's analysis dealt with shifts in mean price and with multiplicative spreads about the mean, both of which satisfy the LS condition. For this case, Meyer showed that the main results of EU analysis are consistent with the MS and that standard hypotheses such as decreasing absolute risk-aversion may be translated into properties of the MS ranking function.

However, the LS condition does not characterize the entire literature on comparative statics under uncertainty, or even the major contributions of Meyer himself. A number of writers including Eeckhoudt and Hansen (1980), Meyer and Ormiston (1983) and Quiggin and Anderson (1981) have derived strong comparative static results for problems such as price-band stabilization in which the distribution of concern is truncated. More recently, a class of increases in riskiness which includes, as a special case, multiplicative spreads about the mean has been proposed independently by Quiggin (1986, 1988) and Meyer and Ormiston (1989) and shown to have tractable comparative static properties. Quiggin refers to these increases in riskiness as monotone spreads.

[1] This model is usually referred to as the mean-variance model. It is clear that preference expressed in terms of μ and σ can equallly well be expressed in terms of μ and σ^2. However, for the derivation fo the concavity conditions emplyed by Meyer, the MS approach is more convenient.

The object of this note is to show that the Meyer analysis has a straightforward generalization to the case of monotone spreads. The analysis is applied to comparative static problems including a portfolio problem with two risky assets and an insurance problem. The results are, in some respects, stronger than those that have previously been obtained using an EU approach.

I. The Monotone Spread Condition and Preference Representation

Meyer adopts a definition of the LS condition, due originally to William Feller (1966), which states that two cumulative distribution functions $G_1(\bullet)$ and $G_2(\bullet)$ differ only by location and scale if for $\beta > 0$,

(1) $$G_1(x) = \alpha + \beta G_2(x) \; \forall \; x$$ [2]

The LS condition is satisfied in many important models of choice under uncertainty, such as those encompassed by the general class of objective functions proposed by Gershon Feder (1977). However, important problems of choice such as that of the optimal level of insurance in the presence of a deductible do not satisfy the LS condition. Moreover, when comparative static analysis is undertaken, the LS condition is highly restrictive since the only changes in risk which can be considered are shifts in the mean and multiplicative spreads about the mean.

A more general condition which has proved useful in comparative static analysis is the monotone spread condition. The concept is best modelled by considering a random variable, not as a cumulative distribution function, but as a mapping y from a state space $\Omega = [0,1]$ to the outcome space $\Re$. We will be concerned in particular with the class of mappings Y_Δ such that $\omega_1 > \omega_2 \Rightarrow y(\omega_1) \geq y(\omega_2)$. Such a mapping may simply be regarded as the inverse of the cumulative distribution function for a continuously distributed random variable.

[2] A minor weakness of this definition is that degenerate distributions (those yielding a given outcome with certainty) cannot be included except in the trivial case when all distributions are degenerate. Thus certainty equivalents, and risk premiums cannot be discussed. It is desirable to modify the definition to require only $\beta \geq 0$. A simple limiting argument ensures that no loss of generality is involved.

Two variables $y_1, y_2 \in Y_\Delta$ are related by a mean-preserving monotone spread if $E[y_1] = E[y_2]$ and $y_1(\omega) - y_2(\omega)$ is a monotone increasing function of ω. Thus, in moving from y_1 to y_2, the bad states get worse and the good states get better. Moreover we may write

(2) $$y_2 = y_1 + \varepsilon, \text{ where } \varepsilon \in Y_\Delta \text{ and } E[\varepsilon] = 0$$

Special cases of the monotone spread relationship include the case where y_1 and y_2 satisfy the LS condition and the case when y_1 is a truncation of y_2.

Quiggin (1986, 1988) and Meyer and Ormiston (1989) show that in typical comparative static problems, such as the standard two asset portfolio problem and the problem of firm output under uncertainty, the optimal action becomes less risky when the relevant random variable undergoes a monotone spread. For example, in the firm problem, output falls when the price undergoes a monotone spread. It is natural, therefore, to investigate distributions which are related by a combination of mean shifts and monotone spreads (and contractions).

II. Preference Representation and Comparative Statics

The simplest extension of the Meyer analysis arises from families of random variables of the form $\mu + y + \lambda\varepsilon$, where $y, \varepsilon \in Y_\Delta$ and $E[y] = E[\varepsilon] = 0$. Consider preferences of the form

(3) $$V(\mu, \lambda) = \int_\Omega U(\mu + y(\omega) + \lambda\varepsilon(\omega))\, d\omega$$

It is easy to obtain a direct extension of the properties derived by Meyer

Proposition 1: Let the choice set consist of random variables of the form $\mu + y + \lambda\varepsilon$, where $y, \varepsilon \in Y_\Delta$ and $E[y] = E[\varepsilon] = 0$. Then Meyer's Properties 1-5 and 7 hold for the preference functional $V(\mu, \lambda)$.

PROPERTY 1 $V_\mu \geq 0$, *if and only if* $U' \geq 0$

PROPERTY 2 $V_\lambda \leq 0$, *if and only if* $U'' \leq 0$

PROPERTY 3 $S(\lambda, \mu) \geq 0$ *if and only if* $U' \geq 0$, $U'' \leq 0$

PROPERTY 4 *V is a concave function of σ and μ if and only if U is concave*

PROPERTY 5 $\partial S(\lambda, \mu)/\partial\mu \leq (=, \geq)$ 0 *if and only if U displays decreasing (constant., increasing)absolute risk aversion*

PROPERTY 7 *If U^1 is a concave transform of U^2, $S^1(\lambda, \mu) \geq S^2(\lambda\, \mu)$ $\forall\, \lambda, \mu$*

Proof:

We have

(4) $$V_\mu = \int_\Omega U'(\mu + y(\omega) + \lambda\varepsilon(\omega))\, d\omega$$

(5) $$V_\lambda = \int_\Omega U'(\mu + y(\omega) + \lambda\varepsilon(\omega))\, \varepsilon(\omega)\, d\omega$$

Clearly V_μ is positive whenever U' is. For V_λ, observe that if U" < 0, U' is monotone decreasing and ε is monotone increasing with $E[\varepsilon] = 0$.

The special case examined by Meyer is that where y is identically zero. However Meyer's proofs of 4, 5 and 7 only make use of the fact that ε is increasing in ω, which is true by the definition of Y_Δ. Hence Meyer's proofs carry over directly. A detailed proof of Property 4 is given in the Appendix. The reader is invited to compare them with the proofs given by Meyer. ■

The only one of Meyer's Properties that does not carry over exactly is Property 6, stated as

PROPERTY 6 $\partial S(t\sigma, t\mu)/\partial t \leq (=, \geq)$ 0 *if and only if U displays decreasing (constant, increasing)absolute risk aversion*

Before considering the extension of this property, it is important to note that it is not strictly correct as stated. The coefficient of relative risk aversion is only defined for positive values of wealth. Hence Meyer's result is meaningful only when $\mu + \min(\varepsilon) \geq 0$. In particular, this means that $\partial V(t\sigma, t\mu)/\partial t \geq 0$, $\forall$ t. Thus, in this case, there is no interior solution to the problem of choosing an optimal t. This is the firm's problem in the case of constant scale returns, or the general control problem of Feder when $A = \phi(x) = 0$ and φ is linear in x.

In the case where $\mu \geq 0$, $\mu + \min(x) \leq 0$, it is straightforward to show that $S(t\sigma, t\mu) = 0$

for t=0, and that $\partial S(t\sigma, t\mu)/\partial t$ is positive, at least in some neighborhood of t = 0, provided U'' ≤ 0. The second-order condition for a unique global solution to the control problem is precisely that $\partial S(t\sigma, t\mu)/\partial t \geq 0, \forall$ t.

The basic problem for an extension of Property 6 to preferences over $(t\lambda, t\mu)$ is that we obtain a term in $(t\mu + t\lambda\varepsilon)/(\mu + y + \lambda\varepsilon)$. In the special case considered by Meyer, this cancels out. More generally, this term is increasing in ω. This yields

PROPERTY 6': *Assume* $\mu+\varepsilon(0) \geq 0$. *Provided relative risk-aversion is non-increasing, so is* $\partial S(t\lambda, t\mu)/\partial t$.

Proof: See Appendix.

Alternatively, we may use t to index variables of the form $(t\mu + ty + t\lambda\varepsilon)$. In this case, Meyer's argument carries over exactly:

PROPERTY 6'': *Assume* $\mu+\varepsilon(0) \geq 0$. $\partial S(t\mu + ty + t\lambda\varepsilon)/\partial t \leq (=, \geq)\ 0$ *if and only if U displays decreasing (constant, increasing)absolute risk aversion*

Proof: As in Meyer

In addition to the results derived by Meyer, it is now possible to consider the impact of changes in y. Suppose that we write the variable as $V(\mu + y + \lambda\varepsilon)$ and consider the impact on $V(\mu, \lambda)$ of a monotone spread in y. Since both y and ε are increasing in ω, a monotone spread in y has effects similar to those of an increase in λ. Thus we would expect V_λ and $S(\lambda, \mu)$ to increase. Let the monotone spread be parametrized by τ

PROPERTY 8: *Given decreasing absolute risk aversion,* $\partial V_\lambda/\partial\tau \geq 0$

PROPERTY 9: *Given decreasing absolute risk aversion,* $\partial S/\partial\tau \geq 0$

Proof: See Appendix.

The extension derived above may be seen as unsatisfactory in two ways. First, in order for the translation between EU and MS to be complete, it would be desirable for the analysis to be expressed in terms of (μ, σ) rankings rather than (μ, λ). Second, it would be desirable to consider more general choice sets, without the linearity restrictions imposed above. This may be done as follows. Let the pair (μ, σ) index a set of random variables $y(\bullet\, ; \mu, \sigma) \in Y_\Delta$ in such a way that for any μ, μ', σ, σ'

(6) $$y(\bullet\, ; \mu, \sigma) = \mu\text{-}\mu' + y(\bullet\, ; \mu', \sigma)$$

and, whenever $\sigma' \geq \sigma$, $y(\bullet\, ; \mu, \sigma)$ and $y(\bullet\, ; \mu, \sigma')$ are related by a mean-preserving monotone spread.

Now, observe that $\partial y/\partial\mu$ is identically equal to 1 and $\partial y/\partial\sigma \in Y_\Delta$, $E[\partial y/\partial\sigma] = 0$ Hence,

(7) $$V(\mu, \sigma) = \int_\Omega U(y(\omega; \sigma, \mu))\, d\omega$$

(8) $$V_\mu = \int_\Omega U'(y(\omega; \sigma, \mu))\, d\omega$$

(9) $$V_\sigma = \int_\Omega U'(y(\omega; \sigma, \mu))\, \partial y/\partial\sigma\, d\omega$$

These are the same as the derivatives obtained above, with $\partial y/\partial\sigma$ taking the place of ε. Since ε is an arbitrary element of Y_Δ with $E[\varepsilon] = 0$, the analysis of Proposition 1 carries over in full. This yields

Proposition 2: Let the choice set consist of random variables of the form $y(\bullet; \mu, \sigma) \in Y_\Delta$ related by mean shifts and monotone spreads. Then Meyer's Properties 1-5 and 7 hold for the preference functional $V(\mu, \sigma)$.

Proof: Follows from Proposition 1 ■

It remains to consider Property 6. In the special case considered by Meyer, the variables $y(\sigma, \mu)$ satisfy the linearity condition $y(\sigma, \mu) = (1/t)y(t\sigma, t\mu)$. In general this identity will not hold. However, some results may be obtained by confining attention to preferences satisfying decreasing absolute risk aversion. Suppose that whenever $t > 1$, $(1/t)y(t\sigma, t\mu)$ third stochastically dominates $y(\sigma, \mu)$. Then, for an individual with constant relative risk-aversion $(1/t)y(t\sigma, t\mu)$ is preferred to $y(\sigma, \mu)$ and $y(t\sigma, t\mu)$ is preferred to $ty(\sigma, \mu)$. Since $S(\sigma, \mu)$ is constant along the path from $y(\sigma, \mu)$ to $ty(\sigma, \mu)$, this suggests that $S(\sigma, \mu)$ should be decreasing along the path from $y(\sigma, \mu)$ to $y(t\sigma, t\mu)$. This result will hold *a fortiori* for the case of decreasing relative risk aversion. Conversely, if whenever $t > 1$, $y(\sigma, \mu)$ third stochastically dominates $(1/t)y(t\sigma, t\mu)$, $S(\sigma, \mu)$ will be increasing in t for constant and increasing relative risk aversion.

III. Applications

Meyer (1987) showed that, in the standard control problem studied by Feder, there is a single random variable of interest, and the outcome is specified as a positive linear function of this random parameter. Examples include the problem of the firm under uncertainty studied by Sandmo and the problem of liquidity preference (Tobin). The generalization offered in Proposition 1 permits the analysis of problems involving two random variables. For example, the portfolio problem may be extended to cover the case of risky base wealth, provided both base wealth and the risky asset are increasing in ω. An alternative way of looking at this problem is that the individual is faced with a choice between two risky assets, one of which is riskier than the other. Let wealth be denoted by W_0, let r_1 and r_2 be the returns on the risky assets, and let α be the proportion of wealth invested in the riskier asset (asset 2). Then the problem may be placed in the framework set out above with

$$\mu = W_0(\alpha E[r_2] + (1-\alpha)E[r_1]) \tag{10}$$

$$\lambda = \alpha \tag{11}$$

$$\varepsilon = W_0[(r_2 - E[r_2]) -(r_1 - E[r_1])] \tag{12}$$

$$y = W_0(r_1 - E[r_1]) \tag{13}$$

The first-order condition is

$$S(\lambda, \mu) = W_0(E[r_2] - E[r_1]) \tag{14}$$

The second-order condition for a unique optimum is

$$V_{\lambda\lambda} + W_0(E[r_2] - E[r_1])V_{\mu\lambda} < 0 \tag{15}$$

The following are a sample of the results that may be obtained by applying the Properties derived above.

(1) Given decreasing absolute risk aversion, an increase in $E[r_2]$ implies an increase in the optimal value α^*. (Follows from (13) and Property 6).

(2) Given decreasing absolute risk aversion, a multiplicative increase in the risk differential ε implies a reduction in the optimal value α^* (Follows from (13) and Property 4)

(3) Given decreasing absolute risk aversion, an increase in the riskiness of both assets (leaving ε unchanged) implies a reduction in the optimal value α^* (Follows from (13) and Property 9).

(4) A more risk-averse individual will have a lower value of α^* (Follows from (13) and Property 7)

Note that the general two risky asset problem, without the assumption that $y, \varepsilon \in Y_\Delta$ is intractable in the absence of strong assumptions on U (Ross).

The problem of the firm under uncertainty may similarly be extended to the case of uncertain fixed costs. This extension is most appealing in the case where uncertainty relates to input costs, since it is likely that states of the world where fixed costs are high will also be states of the world where variable costs are high.

Another extension may be made to the case of decision-making under insurance. Suppose that the individual is to undertake a risky activity with return $\alpha\theta$, where $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ is the control variable for some $\underline{\alpha}$, $\overline{\alpha} > 0$ and $\theta \in Y_\Delta$ is a random variable. The problem is to consider the impact of some form of insurance, γ, such that $-\gamma \in Y_\Delta$ and $\alpha\theta + \gamma \in Y_\Delta$. The requirement that $-\gamma \in Y_\Delta$ means that the insurance variable is risk-reducing and the requirement that $\alpha\theta + \gamma \in Y_\Delta$ means that the individual can never have more than full insurance. Note that since γ depends on the (exogenous) state of the world and not on the observed outcome $\alpha\theta$, problems of moral hazard do not arise.

Insurance is not taken to be a choice variable in the analysis. Thus the results may be interpreted as the impact of some form of publicly provided insurance, such as disaster relief. Alternatively, the analysis yields results on the optimal choice of α, conditional on the prior choice of some insurance package γ.

The problem fits into the framework developed here with

(16) $$\mu = E[\alpha\theta + \gamma]$$

(17) $$\lambda = \alpha - \underline{\alpha}$$

(18) $$\varepsilon = \theta$$

(19) $$y = \underline{\alpha}\theta + \gamma - E[\alpha\theta + \gamma]$$

The first-order condition is simply

(20) $$S(\lambda, \mu) = E[\theta]$$

Results analogous to those derived above for the portfolio may be obtained in this case. However, the most interesting result arises from Property 9. Let an increase in the level of insurance be defined as a monotone spread in $-\gamma$. Then

Proposition 3: An increase in insurance implies an increase in the optimal level of risk-taking α^*.

Proof: Follows from (20) and Property 9.

This result appears to be more general than any that has been obtained in the EU literature on insurance. This provides support for Meyer's (p426) suggestion that the two-parameter approach 'can also allow the investigator to recognize results which might not be obvious in more complicated frameworks.'

The extension to the case of two random variables extends the power of Meyer's two-parameter approach. The more general approach proposed in Proposition 2 yields a further increase in power. Consider a general control problem of the form

(16) $$\mathrm{Max}_{\alpha}\, V = E[U(\phi(\omega, \alpha, W))]$$

where α is a control variable (assumed to take positive values), ω is the state of the world, $W \in Y_{\Delta}$ is stochastic initial wealth, and ϕ is a function mapping actions and states into wealth levels.

This may be fitted into the framework outlined above, provided the following conditions are satisfied

(Φ.1) $\partial\phi/\partial\omega > 0$,

(Φ.2) $\partial\phi/\partial W > 0$

(Φ.3) $\partial^2\phi/\partial\omega\partial\alpha \geq 0$

(Φ.4) $\partial^2\phi/\partial W\partial\alpha \geq 0$

Condition (Φ.1) ensures that $\phi \in Y_\Delta$. Condition (Φ.3) ensures that the standard deviation σ of ϕ is increasing in α and that increases in σ represent monotone spreads in ϕ. Thus, these conditions are sufficient to ensure that the choice problem may be represented in the framework presented here, with first-order condition

$$\mu = E[\phi]$$

$$\sigma = \{E[\phi^2] - \mu^2\}^{0.5}$$

and first-order condition

$$V_\mu E[\partial\phi/\partial\alpha] + V_\sigma E[(\phi - \mu)\, \partial\phi/\partial\alpha\,]/\sigma^2 = 0$$

or

$$S(\sigma, \mu) = \sigma^2 E[\partial\phi/\partial\alpha] / E[(\phi - \mu)\, \partial\phi/\partial\alpha\,]$$

IV. Concluding Comments

By expressing a range of EU comparative static results in terms of a simple two-moment model, Meyer's approach simplifies the analysis and exposition of choice problems under uncertainty. In the present paper, Meyer's results have been extended to a larger class of comparative static problems.

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Proof of Proposition 1: For Property (4)

$$V_{\lambda\lambda} = \int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\varepsilon^2(\omega)\, d\omega \leq 0$$

$$V_{\mu\mu} = \int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\, d\omega \leq 0$$

$$V_{\mu\lambda} = \int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\varepsilon(\omega)\, d\omega \quad 0$$

It is necessary to prove that $V_{\mu\mu}V_{\lambda\lambda} - V_{\lambda\mu}{}^2 \geq 0$

Choose ω^* such that

$$\varepsilon(\omega^*)\int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\, d\omega = \int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\varepsilon(\omega)\, d\omega$$

or

$$\int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))(\varepsilon(\omega) - \varepsilon(\omega^*))\, d\omega = 0$$

The integrand of this expression changes sign once from positive to negative, so

$$\int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\varepsilon(\omega)(\varepsilon(\omega) - \varepsilon(\omega^*))\, d\omega \leq 0$$

Rewriting this as

$$\int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\varepsilon^2(\omega)\, d\omega \geq \varepsilon(\omega^*)\int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\varepsilon(\omega)d\omega$$

and multiplying both sides by $V_{\mu\mu} = \int_\Omega U''(\mu + y(\omega) + \lambda\varepsilon(\omega))\, d\omega$ gives the desired result.

This is simply Meyer's proof with $\varepsilon(\omega^*)$ substituted for x^*. A similar substitution works in the proofs of Properties 5 and 7.

Property 6'

$$\partial S/\partial t\,|_{t=1} = \{- \int_\Omega U'\, d\omega \int_\Omega U''\varepsilon(\mu + \lambda\varepsilon)\, d\omega + \int_\Omega U'\, \varepsilon\, d\omega \int_\Omega U''(\mu + \lambda\varepsilon)\, d\omega\}/\{\int_\Omega U' d\omega\}^2$$

Now the coefficient of relative risk aversion is

$K(\mu + y + \lambda\varepsilon) = [-U''/U'](\mu + y + \lambda\varepsilon)$

Let

$\theta = (\mu + \lambda\varepsilon)/(\mu + y + \lambda\varepsilon)$ Note that $K\theta$ is a decreasing function of ω

Now

$\partial S/\partial t = \{ \int_\Omega U' \, d\omega \int_\Omega U'\varepsilon K\theta \, d\omega - \int_\Omega U' \, \varepsilon \, d\omega \int_\Omega U'K\theta \, d\omega\}/\{\int_\Omega U'd\omega\}^2$

Choose ω^* such that $\varepsilon(\omega^*)\int_\Omega U' \, d\omega = \int_\Omega U' \, \varepsilon \, d\omega$. Then the numerator of $\partial S/\partial t$ is equal to

$\int_\Omega U' \, d\omega \int_\Omega U'(\varepsilon-\varepsilon(\omega^*)) \, K\theta \, d\omega$

which is negative whenever K, and hence $K\theta$, is decreasing ■

PROPERTY 8

Let $\partial y/\partial\tau = \eta \in Y_\Delta$

$\partial V_\lambda/\partial\tau = \int_\Omega U'' \, \eta \, \varepsilon \, d\omega$

Both η and ε are monotone increasing in ω with $E[\varepsilon] = E[\eta] = 0$. Hence the argument used by Meyer and Ormiston (1989, Proposition 2) and Quiggin (1986, 1991) establishes the desired result.

PROPERTY 9

$\partial S/\partial\tau \,|_{t=1} = [- \int_\Omega U' \, d\omega \int_\Omega U''\varepsilon \, \eta \, d\omega + \int_\Omega U'' \, \eta \, d\omega \int_\Omega U'\varepsilon \, d\omega]/\{\int_\Omega U'd\omega\}^2$

Choose ω^* such that $\varepsilon(\omega^*)\int_\Omega U' \, d\omega = \int_\Omega U' \, \varepsilon \, d\omega$. Then the numerator of $\partial S/\partial t$ is equal to

$-\int_\Omega U' \, d\omega \int_\Omega U''(\varepsilon-\varepsilon(\omega^*)) \, \eta \, d\omega$

Hence the argument used by Meyer and Ormiston (1989, Proposition 2) and Quiggin (1986, 1991) establishes the desired result.

PROPERTY 6*

$\partial S/\partial t \,|_{t=1} = \{- \int_\Omega U'd\omega \int_\Omega U''\partial y/\partial\sigma(\mu + \partial^2 y/\partial\sigma\partial t) \, d\omega + \int_\Omega U' \, \partial y/\partial\sigma \, d\omega \int_\Omega U''(\mu + \partial^2 y/\partial\sigma\partial t)d\omega\} / \{\int_\Omega U'd\omega\}^2$

By the argument used for property 6', we need to look at $(\mu + \partial^2 y/\partial\sigma\partial t)/y$. If this is decreasing (increasing) in ω and the coefficient of relative risk aversion is decreasing (increasing) then we get the desired result.