INVARIANCE OF THE EFFICIENT SET - COMMENT

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Quiggin, J. (1992), 'Invariance of the efficient set: comment', *Journal of Economic Behavior and Organization.* 19(1), 119–21.

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In a recent paper, Chew and Zilcha (1990) show that for any set of cumulative distribution functions, the efficient subset generated by the class of generalized expected utility functionals which preserve risk-aversion (in the sense of second stochastic dominance) is the same as the efficient subset generated by the much smaller class of concave expected utility functionals.

The purpose of this note is to show that this result holds in a much broader context. Let $\mathcal{Y}$ be a set with elements denoted y and $\mathcal{V}$ a set of mappings from $\mathcal{Y}$ to $\Re$ with elements denoted v. Let $\mathcal{P}$ be the set of pre-orderings of $\mathcal{Y}$. An element of $\mathcal{P}$ may be regarded as a subset of $\mathcal{Y}^2$. For any $\boldsymbol{P}$ $\mathcal{P}$, the associated strict preference relationship will be denoted by $\boldsymbol{P}$.

For any pre-ordering $\boldsymbol{P}$ $\mathcal{P}$, define a dual subset $\boldsymbol{P^*}$ of $\mathcal{V}$ such that

(1) $$\boldsymbol{P^*} = \{v \quad \mathcal{V} : y_1 \boldsymbol{P} y_2 \qquad v(y_1) \geq v(y_2)\}$$

Similarly, for any subset $\boldsymbol{V}$ of $\mathcal{V}$, define a pre-ordering $\boldsymbol{V^*}$ $\mathcal{P}$ such that

(2) $$y_1 \boldsymbol{V^*} y_2 \Leftrightarrow v(y_1) \geq v(y_2) \qquad v \quad \boldsymbol{V}$$

It is apparent that the larger is $\boldsymbol{P}$ (or $\boldsymbol{V}$), the smaller is $\boldsymbol{P^*}$ (or $\boldsymbol{V^*}$). More importantly, given an initial pre-ordering $\boldsymbol{P}$, it possible to apply both (1) and (2) to obtain a pre-ordering ($\boldsymbol{P^*}$)*. Clearly ($\boldsymbol{P^*}$)* $\boldsymbol{P}$. Similarly given any subset $\boldsymbol{V}$ of $\mathcal{V}$ two iterations of the process yield a larger ($\boldsymbol{V^*}$)* $\boldsymbol{V}$.

The relation between $\boldsymbol{P}$, $\boldsymbol{P^*}$ and ($\boldsymbol{P^*}$)* is obviously similar to standard duality relationships. Given a duality interpretation, it is natural to adopt the notation $\boldsymbol{P^{**}}$ for ($\boldsymbol{P^*}$)* and to refer to $\boldsymbol{P^{**}}$ as the double dual of $\boldsymbol{P}$. When

P** = ***P***, ***P*** will be referred to as maximal, and similarly for ***V***. It may be shown (Quiggin 1990) that for any ***P***, ***P**** and ***P***** are maximal, and, for any ***V***, ***V**** and ***V***** are maximal.

The notion of maximality is central to the argument presented here. The result derived by Chew and Zilcha depends solely on the fact that the partial order associated with second stochastic dominance is maximal in expected utility theory. Numerous other maximal orderings arise in expected utility, notably including first and higher degrees of stochastic dominance. Maximal classes of utility functions include those which display decreasing absolute risk aversion.

Finally, let $\Gamma = 2^{Y}$ be the power set of the space . Then to each partial order ***P*** , there corresponds a mapping $\phi : \Gamma \rightarrow \Gamma$, such that for $S \quad \Gamma$

(3) $$\phi_{\boldsymbol{P}}(S) = \{y \in S : / \ y' \quad S, y' \boldsymbol{P} y\}.$$

The set $\phi_{\boldsymbol{P}}(S)$ is the *efficient subset* of S with respect to the partial order ***P*** (Peleg and Yaari 1975). Since it is clearly possible to recover the partial ordering ***P*** from a knowledge of ϕ (consider the values of ϕ on all two-element subsets of) there is a 1-1 correspondence between ***P*** and ϕ. Similarly, for any ***V*** **,** it is possible to define the efficient subset

(4) $$\phi_V(S) = \{y \quad S : / \ y' \in S, v(y') \geq v(y) \quad v \in V\}$$

There is not, in general, a 1-1 correspondence between ***V*** and ϕ_V. However, any ϕ_V defines a unique maximal ***V***.

P* (and hence ***P***** and ***V*****) will depend on the choice of and hence should strictly be referred to as the dual (or double dual) with respect to . In particular, in the theory of choice under uncertainty it is possible to consider, in addition to the expected utility functionals, a number of possible generalizations. The notation $\boldsymbol{P}_V$* will be used to make the choice of explicit.

It is clear that the larger is the larger is $\boldsymbol{P}_V$*, and hence the smaller is $\boldsymbol{P}_V$**. On the other hand, it is always true that $\boldsymbol{P}_V$** $\boldsymbol{P}$. In particular, if $\boldsymbol{P}$ is maximal with respect to some $_1$**,** $\boldsymbol{P}_{V_1}$** **=** $\boldsymbol{P}$. Hence for any larger $_2$, $\boldsymbol{P}_{V_1}$** $\boldsymbol{P}_{V_2}$** $\boldsymbol{P}$ = $\boldsymbol{P}_{V_1}$**. These remarks are sufficient to prove:

Proposition 1: Let $_2$ $_1$ and let $\boldsymbol{P}$ be maximal with respect to $_1$. Then $\boldsymbol{P}$ is maximal with respect to $_2$.

In particular, let $_1$ be the expected utility functionals, let $_2$ be any class of generalized functionals and let $\boldsymbol{P}$ be any of the usual stochastic dominance orderings in EU theory. Then this result states that the efficient sets generated by the class of functionals in $_2$ which preserve $\boldsymbol{P}$ are precisely the same as those obtained using EU theory. This result, for the special case of second stochastic dominance, is proved by Chew and Zilcha (1990).

The requirement that $\boldsymbol{P}$ be maximal is essential for this extension to work. The result will not hold if, for example, $\boldsymbol{P}$ is the relation corresponding to the requirement that a certain outcome should be preferred to a random variable having the same mean. As is shown by Zilcha and Safra (1989), the class of functionals preserving this relationship in general models yield efficient sets larger than those for the set of EU functionals preserving this relationship (namely the concave ones). An explicit characterization of the efficient sets for the rank-dependent expected utility model (Quiggin 1982, Yaari 1987) is given by Quiggin (1990).

The result derived here has both theoretical and practical implications.

The theoretical implications relate to the extension of EU comparative static results to the case of general preference functionals and are explored in Quiggin (1990). Practical implications arise from the extensive use of stochastic dominance analysis in fields such as project evaluation. Typically, a large set of projects is narrowed by excluding all projects which are dominated in the sense that there exists an alternative project preferred by all EU maximizers whose preferences obey some partial order. Provided this partial order is maximal for EU, the result given here shows that the validity of the analysis does not depend on the assumption that the individual evaluating the project is an EU-maximizer.

In summary, the result obtained by Chew and Zilcha is one example of a much more general principle. The concept of a maximal preference ordering may prove useful in exploring the relationships between various theories of choice under uncertainty and their implications for problems such as portfolio choice.

References

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