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*Abstract*

*Safra and Zilcha (1988) have examined the behavior of efficient sets in generalized models. The object of this note is to generalize their results. It is shown that, for the class of continuous preference functionals preserving any continuous partial ordering on the space of cumulative distribution functions, the efficient subset of any given set of prospects are simply those which are not dominated under the original partial ordering.*

# EFFICIENT SETS WITH AND WITHOUT THE EXPECTED UTILITY HYPOTHESIS - A GENERALIZATION

The notion of stochastic dominance has been one of the central features of the Expected Utility theory of choice under uncertainty. An important application of stochastic dominance analysis is the derivation of efficient subsets of a set of risky prospects (those which are not stochastically dominated by any available alternative). In an investment or decision problem attention can be confined to the efficient set. The theory of efficient sets in the expected utility model is well-developed. A number of generalizations of expected utility have been proposed, including weighted utility theory (Chew 1983), the theory of local utility functions (Machina 1982) and rank-dependent expected utility (Quiggin 1982, Yaari 1987).

Safra and Zilcha (1988) have examined the behavior of efficient sets in generalized models. Their analysis begins with the consideration of a range of definitions of risk-aversion in terms of partial orderings on the space of cumulative distribution functions. Under expected utility, these definitions are all equivalent, in the sense that they characterize the same class of utility functions (namely, the concave increasing functions) and hence yield the same efficient sets. Safra and Zilcha then consider the implications of dropping the expected utility hypothesis, and admitting arbitrary continuous preference functionals on the space of cumulative distribution functions. They show that, in this case, the different definitions of risk-aversion are no longer equivalent. Further, the efficient sets they yield are essentially equivalent to the original partial ordering.

The object of this note is to generalize this result. It is shown that, for the class of continuous preference functionals preserving any continuous partial ordering on the space of cumulative distribution functions, the efficient subset of any given set of prospects are simply those which are not dominated under the original partial ordering.

## Preference orderings and efficient sets

In order to place the discussion in a more general context, I will depart from the notation of Safra and Zilcha. Let $\mathsf{Y}$ be the set of random variables on some subset of $\Re$, A generic element of $\mathsf{Y}$, denoted $y$, is defined as a Lebesgue measurable mapping from the unit interval [0, 1] into $\Re$. The associated space of cumulative distribution functions is denoted $\mathsf{D}$ and a generic element is denoted $F$. Let $\mathsf{V}$ be a set of mappings from $\mathsf{Y}$ to $\Re$ with generic elements denoted $v$. Let $\mathsf{P}$ be the set of pre-orderings of $\mathsf{Y}$. A generic element of $\mathsf{P}$ is denoted $\boldsymbol{P}$, and the associated strict ordering is denoted $\boldsymbol{P}$

For any pre-ordering $\boldsymbol{P}$ $\mathsf{P}$, define a dual subset $\boldsymbol{P^*}$ of $\mathsf{V}$ such that

(1) $$\boldsymbol{P^*} = \{v \quad \mathsf{V} : y_1 \boldsymbol{P}\, y_2 \quad v(y_1) \geq v(y_2)\}$$

Similarly, for any subset $\boldsymbol{V}$ of $\mathsf{V}$, define a pre-ordering $\boldsymbol{V^*}$ $\mathsf{P}$ such that

(2) $$y_1 \boldsymbol{V^*}\, y_2 \Leftrightarrow v(y_1) \geq v(y_2) \quad v \quad \boldsymbol{V}$$

It is apparent that the larger is $\boldsymbol{P}$ (or $\boldsymbol{V}$), the smaller is $\boldsymbol{P^*}$ (or $\boldsymbol{V^*}$). More importantly, given an initial pre-ordering $\boldsymbol{P}$, it possible to apply both (1) and (2) to obtain a pre-ordering $(\boldsymbol{P^*})^*$. Clearly $(\boldsymbol{P^*})^*$ $\boldsymbol{P}$. Similarly, given any $\boldsymbol{V}$ $\mathsf{V}$, two iterations of the process yield a larger $(\boldsymbol{V^*})^*$ $\boldsymbol{V}$.

In important cases, the inclusions derived above are strict. For example, let $\mathsf{V}_{EU}$ be the space of expected utility functionals, and let $\boldsymbol{P}$ be the pre-ordering corresponding to the simplest definition of risk aversion, namely that a sure thing is preferred to a risky prospect $y$ with the same expectation. The most basic result of expected utility theory is that this condition is satisfied (that is, $v$ $\boldsymbol{P}^*$) if and only if the utility function $U$ is everywhere concave. The theory of second stochastic dominance, developed by Hadar and Russell (1967) shows that, whenever $y_1$ SSD $y_2$, $v(y_1) \geq v(y_2)$ $v$ $\boldsymbol{P}^*$.

The relation between $\boldsymbol{P}$, $\boldsymbol{P^*}$ and $(\boldsymbol{P^*})^*$ is obviously similar to standard duality relationships. Given a duality interpretation, it is natural to adopt the notation $\boldsymbol{P^{**}}$ for $(\boldsymbol{P^*})^*$ and to refer to $\boldsymbol{P^{**}}$ as the double dual of $\boldsymbol{P}$. Further it is natural to consider the case when $\boldsymbol{P^{**}} = \boldsymbol{P}$. In this case ***P*** (or ***V***) will be referred to as maximal. It is straightforward to show (Quiggin 1990) that, for any ***P***, ***P**** is maximal, and for any ***V***, ***V**** is maximal. It should be noted that ***P**** (and hence ***P**** and ***V****) will depend on the choice of V and hence should strictly be referred to as the dual (or double dual) with respect to V.

Finally, let $\Gamma = 2^{Y}$ be the power set of the space Y. Then to each partial order ***P*** , there corresponds a mapping $\phi : \Gamma \rightarrow \Gamma$, such that for $S \quad \Gamma$

(3) $$\phi_{P}(S) = \{y \quad S: \not\ y' \quad S, y' P\, y\}.$$

The set $\phi_P(S)$ is the *efficient subset* of *S* with respect to the partial order ***P*** (Peleg and Yaari 1975). Since it is clearly possible to recover the partial ordering ***P*** from a knowledge of $\phi$ (consider the values of $\phi$ on all two-element subsets of Y) there is a 1-1 correspondence between ***P*** and $\phi$. Similarly, for any ***V*** V, it is possible to define the efficient subset

(4) $$\phi_V(S) = \{y \quad S: \not\ y' \in S, v(y') \geq v(y) \quad v \in V\}$$

There is not, in general, a 1-1 correspondence between ***V*** and $\phi_V$. However, any $\phi_V$ defines a unique maximal ***V***.

It is now possible to prove

**Proposition 1:** Let ***P*** be a closed ordering preserving first stochastic dominance, and let V = {$v$: Y → ; $v$ continuous}.

Then ***P**** = ***P***.

**Corollary**: Let ***V*** = ***P****. Then, for any *S*, $\phi_V(S) = \phi_P(S)$.

**Proof:** Choose $y_1$, $y_2$ unrelated by ***P***. That is, neither $y_1$ ***P*** $y_2$ nor $y_2$ ***P*** $y_1$ holds. If no such $y_1$, $y_2$ exist, ***P*** is a total ordering and the result is trivial.

Let $A = \{y{:}y_1\ P\ y\}$ $B = \{y{:}y\ P\ y_2\}$. By the transitivity of ***P*** and the fact that $y_1$, $y_2$ are unrelated by ***P***, A and B are disjoint. I claim that we can define a continuous function $v$ V such that

(i) $v = 0$ on $A$, $v = 1$ on $B$

(ii) $y\ \boldsymbol{P}\ y' \quad v(y) \geq v(y')$

The proof is essentially the same as that of Urysohn's lemma given by Dugundji (1966, p 147). We first show that for all elements $r$ of the set $R$ of rationals of the form $k/2^n$ we can define an open set $N(r)$ such that

(iii) $A \quad N(r)$ and $N(r) \cap B =$

(iv) $r > r' \quad \overline{N(r')} \quad N(r)$

(v) $y \in U(r)$, $y\ \boldsymbol{P}\ y' \quad y' \quad U(r)$

Let $B^C$ be the complement of $B$. By normality there exists an open set $N$ containing $A$ and having its closure contained in $B^C$.

Define

(5) $$N(0) = \{y \in \quad ; \quad y' \in N, y' \boldsymbol{P} y\}$$

Because $\boldsymbol{P}$ preserves first stochastic dominance and $N$ is open, $N \quad N(0)$. Because $\boldsymbol{P}$ is closed, the closure of $N(0)$ is contained in $B^C$. The remainder of the construction is a straightforward induction, following Dugundji.

It is now possible to define $v$. Set $N(1) =$ and:

(6) $$v(y) = \inf \{r : y \quad N(r)\}$$

Define $\boldsymbol{P}_1$ by

(7) $$y\ \boldsymbol{P}_1\ y' \text{ iff } v(y) \geq v(y').$$

Then it is clear that $v$ has the desired properties and that $\boldsymbol{P}_1$ is a closed partial ordering preserving $\boldsymbol{P}$ and such that $y_1\ \boldsymbol{P}_1\ y_2$ . Since we can clearly define $\boldsymbol{P}_2$ such that $y_2\ \boldsymbol{P}_2\ y_1$, it is apparent that neither $(y_1, y_2)$ or $(y_2, y_1)$ is an element of $\boldsymbol{P}^{**}$. Hence $\boldsymbol{P}^{**} = \boldsymbol{P}$ as desired. The corollary follows immediately. n

The main result of Safra and Zilcha (1988) may be derived as a special case of Proposition 1. For $\alpha$ [0,1], $F$ , $t$ , they define

(8) $$F_{\alpha}(t) = F((t - \alpha \, \mathrm{E}[F])/(1-\alpha))$$

Consider the partial ordering $\boldsymbol{P}_{\alpha}$ defined by

(9) $$\boldsymbol{P}_{\alpha} = \{(F, F_{\beta}), \beta \quad [\alpha, 1]\}$$

along with the usual join with first stochastic dominance. More formally this join may be written by stating

(10) $$G \, \boldsymbol{P}_{\alpha} \, F \text{ if and only if } \quad \beta \quad [\alpha, 1] \; G \text{ FSD } F_{\beta}$$

Safra and Zilcha (1988) use the term ‘$\alpha$ risk-averse’ to characterize preference functionals $V$ $\boldsymbol{P}_{\alpha}$*. For any set $X$ , $Z_{\alpha}(X)$ is defined to be the efficient set with respect to the class of $\alpha$ risk-averse preference functionals, $\boldsymbol{P}_{\alpha}$*. By Proposition 1, $\boldsymbol{P}_{\alpha}$** = $\boldsymbol{P}_{\alpha}$, so $F \in Z_{\alpha}(X)$ if and only if there is no $G$ $X$, such that $G \, \boldsymbol{P}_{\alpha} \, F$. Safra and Zilcha (1988, Theorem 1) is a constructive proof of this result.

## Concluding Comments

The Expected Utility imposes strong linearity requirements on preference relationships. It is precisely this linear structure which has permitted the derivation of the central results in the EU theory of efficient sets. In these results, linearity is used to extend an original preference relation to its convex hull. It is unsurprising, therefore, that in the absence of any restrictions on preference other than continuity, no such extension is feasible. Most of the alternatives to, and generalizations of, EU theory retain some linear or quasi-linear structure. Some implications for efficient sets are discussed in Quiggin (1990).

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