ON THE OPTIMAL DESIGN OF LOTTERIES

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ABSTRACT

In recent years the Expected Utility model of choice under risk has been generalized to cope with phenomena such as probability weighting. In the present paper, one such generalized approach, the Rank-Dependent Expected Utility model, is applied to the problem of lottery gambling. The model is used to derive an optimal prize structure for lotteries, involving a few large prizes and a large number of small prizes. Other forms of gambling, such as racetrack betting, are discussed in the light of this result.

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INTRODUCTION

The expected utility (EU) theory developed by von Neumann and Morgenstern (1944) has been one of the success stories of modern economics. It has permitted formal analysis of a large and important area of economic behavior which had previously been largely outside the realm of economic theory. Problems such as insurance, price stabilization and portfolio choice have been subjected to rigorous analysis. Meanwhile, the main competing approach, the mean-variance analysis advocated by Markowitz (1959) has been largely abandoned as a tool of economic analysis, although it is still used as a rough and ready guide to portfolio choice. The advantages of the EU approach included a clear axiomatic basis, greater generality and the availability of tools such as the Arrow-Pratt coefficients of risk-aversion.

The EU approach has been fruitful as a source of novel and unexpected hypotheses. These include some ideas which are meaningful only in the context of EU theory, such as the hypothesis of declining absolute risk-aversion, as well as others, such as the idea that full insurance is sub-optimal, which reflect the analytical power of the approach.

Finally, the EU approach has permitted the development of new and powerful analytical tools and concepts. The Arrow-Pratt coefficients of relative and absolute risk-aversion have already been mentioned. The concepts of stochastic dominance (Hadar and Russell 1969) and increasing risk (Rothschild and Stiglitz 1970,1971) are also closely related to the EU framework.

This very real success has, however, been achieved despite the existence of a growing body of theoretical and empirical objections to EU theory. These date back as far as the work of Allais (1953, translated in Allais and Hagen 1979) who rejected the EU independence axiom and produced as a counter-example the well-known Allais paradox. Major theoretical objections were also raised, at the outset, by post-Keynesian writers such as Shackle (1952).

The body of evidence conflicting with EU theory has grown over time. Three main groups of problems may be distinguished. First, there are problems observed in experiments such as those of Ellsberg (1961) and MacCrimmon and Larsson (1979). Second, there are difficulties associated with attempts to construct utility functions using questionnaires. Finally, there are behavioral choices, such as those relating to gambling and the widespread preference for full insurance which appear to conflict with the theory. It is the problem of gambling behavior which will be examined in the present paper.

Despite a number of attempts, most notably that of Friedman and Savage (1948), it has not been possible to produce an adequate explanation of observed gambling behavior within the EU framework. In addition to a discussion of these approaches and the objections which have been made against them, this paper includes a new argument against the Friedman-Savage approach.

The problems associated with EU have led to many attempts to construct alternative theories of behavior under risk such as those of Allais (1953), Handa (1977), Karmarkar (1978, 1979) and Kahneman and Tversky (1979). Until recently, however, none of these approaches has been able to match the sharp predictions yielded by EU theory. More recently, however, generalizations of EU theory have been developed by Machina (1982), Quiggin (1982) and Chew (1983). These models account for many of the observed violations of the predictions of EU theory, such as the Allais paradox (Machina 1983, Quiggin 1985, Segal 1987) while retaining its analytic power in that many of the standard EU analytic concepts, such as risk-aversion can be generalized. The primary approach used here is the rank-dependent expected utility (RDEU) model (Quiggin 1982, Yaari 1987). A number of the results can be carried over to the generalized expected utility framework of Machina (1982).

The object of the present paper is to extend analysis to an area not handled successfully by EU theory, namely the demand for gambling opportunities such as lottery tickets. The Friedman-Savage model of simultaneous gambling and insurance is discussed and criticisms of it reviewed. An even more fundamental flaw in the model, not previously observed, is pointed out. The RDEU model is outlined and compared with Machina's (1982) model. Optimality conditions for a lottery are derived and interpreted for both models. It is shown that the RDEU model provides a good explanation of observed lottery design.

I. GAMBLING AND INSURANCE

One of the first major problems to emerge in EU theory arose from observed behavior in markets involving uncertainty such as those for insurance and gambling. Gambling, as well as being the inspiration for the development of probability theory, is an economically significant aspects of uncertainty. Yet gambling is inconsistent with EU theory unless the very plausible hypothesis of risk-aversion is abandoned, and the coexistence of gambling and insurance purchase is very difficult to fit into the EU framework.

Some types of gambling can be explained easily enough. For example, given the virtual certainty of losing on slot machines in the medium run, it seems reasonable to assume that this activity is undertaken largely for entertainment. Similar explanations may be advanced for participation in various time-consuming gambling games such as bingo. Racetrack betting provides a more complex case. Some betting on horse races may reflect divergent beliefs about the outcome. This is reflected in the large amounts of effort devoted to collecting and analyzing information about the quality of horses, jockeys, tracks etc. On the other hand, there are a large number of bettors who collect no information at all, and bet on an essentially random basis. The problem of racetrack betting will be discussed further below.

In respect of lottery tickets, however, there is no acceptable explanation consistent with risk aversion. The only plausible reason for betting is the chance of winning a large amount of money. The predominance of this motive is confirmed by psychological studies (Walker 1984). Thus, the purchase of lottery tickets by people who are generally risk-averse constitutes a significant problem for EU theory, which is very difficult to explain without resorting to the content-decreasing measure of excluding gambling from the realm of rational behavior.

Friedman and Savage (1948) attempted to explain the coexistence of gambling and insurance using the concept of an *S*-shaped utility function (see Figure 1). The basic idea is that people will be risk averse with respect to changes in a neighborhood of their current wealth level but may be risk-seeking with respect to prospects which may take them into a higher social class. Thus, the utility function may be regarded as concave for low income levels and convex for some higher incomes. The function initially suggested by Friedman and Savage had this form. The third concave segment (at still higher income levels) is a modification introduced as a response to the observation that lotteries typically have multiple prizes whereas convexity throughout the upper range would imply that a lottery with a single large prize would be preferred.

It may be noted that this modification does not in fact account for the observed distribution of prizes in lotteries. Preferences of this kind would imply a desire for a lottery with several prizes of the same value, a pattern which is rarely observed in practice. These preferences cannot account for the typical prize distribution having a few large prizes and a large number of small ones.

FIGURE 1 NEAR HERE

A variation on this theme, expressed in terms of indivisibilities in expenditure, is developed by Ng (1975). He argues that even if the underlying utility function is concave, the existence of large indivisible items of expenditure (eg the purchase of a university education) will lead people to buy lottery tickets. Broadly similar approaches to the problem have been taken by Flemming (1969) and Kim (1973) . None of these approaches appear adequate as explanations of gambling in developed countries with well-developed credit markets where a large proportion of people buy lottery tickets. Even allowing for considerable costs associated with imperfections in credit markets, it is unlikely that these could justify the purchase of lottery tickets with an expected return of about 60 cents for each dollar in outlays.

A number of objections to the Friedman-Savage approach have been made by Machina (1982). The most important is that the observed gambling behavior of individuals does not appear to change radically in response to changes in their initial wealth. However, the utility function in Figure 1 suggests that behavior will be highly sensitive to changes in initial wealth, with only those individuals near the inflexion points displaying propensities to both gamble and insure.

All of the approaches cited above have focused on the value of the outcomes. This is a natural consequence of the use of the EU framework, but it seems far more reasonable to suppose that participation in lotteries has to do with attitudes to probabilities and, in particular, with the placing of a high weight on extremely favorable, low probability events. Adam Smith (1776, pp 164-165) stated:

> That the chance of gain is naturally over-valued, we may learn from the universal success of lotteries. The world neither ever saw, nor ever will see, a perfectly fair lottery; or one in which the whole gain compensated the whole loss; because the undertaker could make nothing by it. In the state lotteries, the tickets are not worth the price which is paid by the original subscribers, and yet commonly sell in the market for twenty, thirty and sometimes forty per cent advance. The vain hope of getting some of the great prizes is the sole cause of this demand. The soberest people scarce look upon it as folly to pay a small sum for the chance of gaining ten or twenty thousand pounds; though they know that even that small sum is perhaps twenty or thirty per cent more than the chance is worth. In a lottery in which no prize exceeded twenty pounds, though in other respects it approached nearer to a perfectly fair one than the common state lotteries, there would not be the same demand for tickets. In order to have a better chance for some of the great prizes, some people purchase several tickets, and others, small shares in a still greater number. There is not, however, a more certain proposition in mathematicks, than that the more tickets you adventure upon, the more likely you are to be a loser. Adventure upon all the tickets in the lottery, and you lose for certain; and the greater the number of your tickets the nearer you approach to this certainty

This passage encapsulates a number of objections which have been made to the EU explanations of gambling and a number of requirements for a successful explanation. In particular, it is necessary that the theory should explain preference for lotteries with a few large prizes, that it should not suggest that people are unaware that the game in which they are participating is not (actuarially) fair, and that it should explain the purchase of one or a few tickets.

Even with the abandonment of the plausible postulate of risk-aversion, EU theory cannot account for the observed behavior of a large segment of the population in undertaking both gambling and insurance. What is needed is a model of choice under uncertainty which takes explicit account of phenomena such as probability weighting which have frequently been recognized in psychological studies of the problem such as those of Edwards (1962). Such a model is developed in the following section.

II. AN OUTLINE OF THE RANK-DEPENDENT MODEL

RDEU theory deals with individual preferences over a set W of outcomes and an associated set Y of prospects, or random variables, taking values in W. For the purposes of this article, W will be assumed to be a connected subset of $\mathbb{R}$, and may be interpreted in terms of wealth levels. A random variable $y \in Y$ may be regarded as a measurable function from some σ-field of events, Ω, to W, and characterised by its cumulative distribution function, $F_y : w \to \Pr\{y \le w\}$. It will be assumed, without loss of generality, that the states of the world, $\omega \in \Omega$, are ordered from worst to best so that $\omega_1 \le \omega_2 \Rightarrow y(\omega_1) \le y(\omega_2)$.

In this paper, attention will be confined to discrete random variables or 'prospects' (those with finite support), although the model is quite applicable to continuous distributions. These may be represented in the form

(1) $\{\boldsymbol{w};\boldsymbol{p}\} = \{(w_1, w_2, \ldots w_N); (p_1, p_2, \ldots p_N)\}$,

where p_j is the probability of outcome w_i, $\sum_{i=1}^{N} p_i = 1$, and the w_i are ordered from worst to best, so that $w_1 \le w_2 \le \ldots \le w_n$. EU theory involves constructing a utility function U on W and setting

(2) $V(\{w;\boldsymbol{p}\}) = \sum_{i=1}^{N} p_i U(w_i)$

The RDEU model includes EU as a special case. In addition to the utility function, it employs a probability weighting function $q: [0,1] \to [0,1]$. The function q is continuous, monotonically increasing and such that $q(0) = 0, q(1) = 1$. This function may be composed with a cumulative distribution function F to yield a weighting function $H = q \circ F$, which possesses the usual properties of a cumulative distribution function. In the present context, where we are mainly concerned with the upper tail of the distribution, it is also convenient to define $q^*(p) = 1 - q(1-p)$.

The RDEU functional is defined as

(3) $$V(\{w;\boldsymbol{p}\}) = \text{iSU}(i{=}1{,}N{,}\ )\ U(w_i)\ h_i(\boldsymbol{p}),$$

where

(4) $$h_i(\boldsymbol{p}) = q(\ \text{iSU}(i{=}1{,}k{,}\ )p_k) - q(\text{iSU}(i{=}1{,}k{-}1{,}\ )p_k)$$

It will be assumed throughout that the utility function U is globally concave, since the arguments against the plausibility of a convex utility function, advanced in the previous section in relation to EU theory, apply with equal force to the RDEU model. By contrast, as is shown below, these arguments do not apply to risk-seeking behavior associated with the possibility that $q^*(p) > p$ for small p.

Assuming that extreme low-probability events are overweighted, the function q will take the shape illustrated in Figure 2, which is clearly reminiscent of the Friedman-Savage utility function. The basic requirement for this condition is that $q'(p)$ should be a U-shaped function convex in p. In order to avoid St. Petersburg paradoxes, it is also necessary to assume that $q'(p)$, and hence $q(p)/p$, are bounded.

(FIGURE 2 NEAR HERE)

It is intuitively apparent that the pattern of probability weighting illustrated in Figure 2 is consistent with both gambling and insurance. The overweighting of outcomes at the lower tail of the distribution reinforces the effects of a concave utility function (risk-aversion in the EU sense) in promoting a demand for insurance against adverse low-probability events. It may be noted, however, that the effects of probability weighting discourage insurance against high probability events. By contrast, the effects of overweighting on events at the upper tail of the distribution counteract the effects of EU risk aversion which discourage low-probability, high-payoff gambles. The extent to which gambles are accepted depends on the relative curvature of the U and q functions. However, it is straightforward to show that for any person with a weighting function of the kind illustrated in Figure 2, there are some actuarially unfair gambles which will increase utility.

III. LOTTERIES WITH A SINGLE PRIZE

The discussion above has indicated that the RDEU model is capable of explaining simultaneous gambling and insurance. However, such an explanation is only valuable if the model can generate sharp and (at least in principle) testable predictions of behavior. While some individual testing has been carried out (see Quizon, Binswanger and Machina 1984 and Quiggin 1981), the most useful tests are those which relate to behavior in actual gambling and insurance markets. Gambling markets are of particular interest because they have previously been outside the scope of effective economic analysis. Thus it is important to see whether the observed structures of gambling markets and, in particular, the gambles offered and accepted are consistent with the predictions of the theory.

In this section, the RDEU model will be applied to the problem of determining the optimal prize structure for a lottery, given that all consumers have identical RDEU preferences. Thus, if the RDEU model is correct, we would expect that the prize structure actually prevailing will be fairly similar to the optimal solution derived here. The consequences of relaxing the assumption of identical consumers will be considered below.

It will be assumed that the operator's return is predetermined, either by government regulation or by a perfectly competitive market, and that the problem is to maximize the perceived net value of the ticket to the potential purchaser. This would appear to correspond fairly closely to the actual situation for lottery-type gambles. Attention is therefore confined to actuarially fair gambles, although the analysis extends fairly easily to the case of a fixed 'rake-off'.

Before proceeding to the general solution of the problem, it is useful to consider two subproblems. An individual's demand for any commodity is determined by their demand curve and the market price. In this context, it is useful to think of the demand curve being determined by the anticipated utility of the (net) prize payouts on the winning tickets, and the price being the anticipated utility loss associated with the losing tickets. Thus, in order to optimize lottery design, the operator should maximize the anticipated gains and minimize the anticipated losses. Given the assumption of actuarial fairness, these problems may be analysed separately.

The simpler of these two subproblems is that of minimizing anticipated losses. Since ticket prices are normally small in relation to total wealth, the utility function U can be approximated by a function linear in the ticket charge c. Units can be chosen so that the required average revenue is 1, and the decision problem becomes

(5) $\quad \text{Min } c\, q(p)$

subject to $cp = 1$

where p is the probability of a losing ticket. This problem is equivalent to minimizing $q(p)/p$ and the first and second order conditions on p are

(6) $\quad pq'(p) - q(p) = 0$

and

(7) $\quad pq''(p) > 0$

respectively. It is easy to show that these conditions will be satisfied at point A in Figure 2, where a ray drawn from the origin is tangent (from below) to the weighting curve.

While data are scanty and not very reliable, the evidence which is available, such as Edwards (1962), suggests that this point will lie somewhere between 0. 75 and 0. 95. The solution derived here has direct application to one class of gambles, namely promotions for a product in which purchasers have some chance of having the purchase price refunded, that is, of getting the product for nothing. Obviously, this involves an effective increase in price, since the promotional funds could equally be applied to an across the board price reduction. The analysis given here suggests that promotions of this kind will be most effective in increasing demand if the proportion of refunds lies between 1 in 4 and 1 in 20. In general, however, the gamble offered in a promotion of this kind will be dominated by a lottery ticket offering a chance of a large positive prize. Hence such gambles will appeal only to consumers who are unable (or perhaps unwilling) to participate in lotteries, or who exhibit an 'isolation' effect, in which the gamble offered by the promotion is assessed without taking account of other available gambles. The fact that promotions of this kind are offered only occasionally supports the view that they are attractive only when some kind of isolation effect comes into play.

The problem of maximizing the anticipated value of the payout is somewhat more complex because of the need to take both probability and risk attitudes into account. If only probability attitudes were considered, the problem would be one of maximizing $q^*(p^*)/p^*$, where p^* is the probability a winning ticket. However, the assumptions behind the function illustrated in Figure 2 mean that this value increases as p^* decreases, converging to $(q^*)'(0)$ as p^* converges to zero. Thus, the optimal solution would be an infinitely large prize with an infinitesimally small chance of winning. Risk aversion offsets this and implies that a finite optimal prize will exist for a lottery with a single prize and a fixed average return.

Expressing the utility function in terms of deviations from initial wealth, the problem may be expressed as

(8) $$\text{Max } U(w)q^*(p^*) + \lambda(1 - wp^*)$$

Deriving the initial conditions and solving for the Lagrangean multiplier λ yields

(9) $$f(wU'(w) , U(w)) = f(p^*q^{*\prime}(p^*), q^*(p^*))$$

The LHS of (9) may be interpreted as a measure of relative risk aversion. It is different from $r_r(w)$, the standard Arrow-Pratt coefficient of relative risk aversion, which refers to risks in a neighborhood of w. The measure given here compares the marginal utility of additional wealth at w h the average utility over the whole range $[0,w]$. Nonetheless, the two are closely related. In particular, for the class of constant relative risk-aversion utility functions $U(w) = w^{\alpha}$, we have

(10) $$r_r(w) = 1 - \alpha \ , \ f(wU'(w) , U(w)) = \alpha, \forall w.$$

A similar interpretation may be given to the RHS. It is a measure of the weight given to a small change in p compared to the weight on the total probability of winning $q^*(p^*)$.

Before considering the properties of this solution, it is necessary to establish that a solution does in fact exist. This may be shown as follows, subject to fairly weak conditions.

First, in order for an increase in the prize to be profitable, it is necessary that the LHS of (9) be greater than the RHS. Given the shape of the weighting function in Figure 2, some actuarially fair gambles are regarded as favorable, so this condition must be satisfied somewhere. However, the concavity of U implies that the LHS is always less than one, and, provided $r_r(w)$ is bounded away from zero, the LHS will be bounded away from 1. However, the boundedness condition on q' implies that the RHS approaches 1 as p^* approaches zero, and hence that (8) must be satisfied for some finite w and p^*.

This discussion also gives a good indication of the comparative static properties of the solution. In general the optimal prize will be larger, the larger is the overweighting of small probabilities and the lower is the level of relative risk-aversion.

IV. THE GENERAL CASE

The most important point to note from the analysis of the previous section is that there is no necessary relationship between the probability of winning which maximizes the anticipated value of the prize and the probability of losing which minimizes the anticipated loss. For plausible values of the coefficients, the former will be very close to zero, while the latter is likely to be significantly less than 1. This provides an intuitive justification for the existence of multiple prizes in lotteries, and, in particular, for the frequently adopted practice of giving a large number of fairly small prizes which do not contribute much to the expected value of the ticket. By reducing the probability of losing from near-certainty to a value of p for which significant underweighting applies, this practice reduces the anticipated loss associated with the ticket.

The rationale for multiple prizes given here must be distinguished from the discussion of this issue by Friedman and Savage (1948). Friedman and Savage justified the final concave segment in their utility function by the observation that lotteries have multiple prizes and hence that people are presumably not risk-preferrers for unbounded wealth levels. However, what they were really concerned about here was the fact that there is a finite optimal bet, and this can be inferred more satisfactorily from the observation that otherwise there would be only one big lottery in any given time-period. The Friedman-Savage theory does not predict multiple prizes unless it is assumed that there is a fixed minimum size for the lottery pool, and even then it would predict a number of identical prizes rather than the observed pattern of a maximum prize and a series of smaller prizes.

With these considerations in mind, it is possible to analyze the general problem of optimal lottery design. It will be assumed that there is a fixed number of tickets N. Provided N is large, this is not a very severe restriction on the generality of the results. For any lottery with say, 1000 tickets at $10 each, a fairly close substitute can be designed having 10,000 tickets at $1 each. There are thus N outcomes each occurring with probability $1/N$. These will be denoted $w_1, \ldots, w_N$, and ordered as in Section 3, from worst to best, so that w_N is the first prize, while w_1 is the maximum loss (in a normal lottery, this will be the price of a ticket). As in equation (7), the weight associated with outcome i will be denoted h_i. The optimization problem must take account of both the constraint on the operator's rake-off and the ordering of the w_i. In order to simplify the former constraint, it will be assumed that the lottery is constrained to break even. The introduction of positive profits does not lead to major changes in the solution. The problem may be formulated as

(11) $$\text{Max} \sum_{i=1}^{N} h_i U(w_i) + \lambda\left(\sum_{i=2}^{N} w_i\right) + \sum_{i=1}^{N} \mu_i (w_i - w_{i-1})$$

where the μ_i are Kuhn-Tucker multipliers.

The first order conditions are

(12) $$h_i U'(w_i) + \lambda + \mu_i - \mu_{i+1} = 0 \qquad \forall i \qquad (\text{setting } \mu_0 = 0)$$

(13) $$\mu_i(w_i - w_{i-1}) = 0, \qquad \mu_i \geq 0, \qquad (w_i - w_{i-1}) \geq 0 \qquad \forall i$$

The interpretation of these conditions is straightforward for the upper tail of the distribution, where (12) can be satisfied with the μ_i equal to zero. There will be a declining sequence of prizes starting at w_N and satisfying the relationship

(14) $$U'(w_i) / U'(w_k) = h_k / h_i .$$

That is, the declining marginal utility of income is exactl,y offset by the increasing weight placed on events at the upper tail of the distribution. This relationship is particularly tractable for the constant relative risk-aversion functions of the form $U(w) = w^{\rho}$, discussed above. The relationship reduces to

$$(15) \qquad w_i/w_k = (h_k/h_i)^{\gamma} \qquad \text{where } \gamma = -(1/1-\rho) = -1/r_r.$$

While the ordering of the w_i is essential for a tractable solution to the problem, there are some difficulties in determining the precise nature of the solution over the range where the λ_i are not necessarily zero. The most obvious solution is one where $w_1 \ldots w_k$ are all equal (and, of course, negative, being equal to the ticket price) and $w_{k+1} \ldots w_N$ are an increasing sequence of prizes satisfying (14). In this case, it would appear that the analysis of the first subproblem can be carried over to show that k/N, the probability of a losing ticket, will be chosen to minimise $q(p)/p$. The next proposition shows that this is, in fact, the optimal solution under natural conditions.

Proposition 1: Let the utility function be U be concave and let q be such that $q'(p)$ is convex in p. Then the optimal prize structure is as described above.

Proof: See Appendix.

V. EVIDENCE FROM OBSERVED LOTTERY DESIGNS

The prize structure described above seems to correspond fairly well to that prevailing in actual lotteries. There are, however, a number of qualifications which should be taken into account. First, in most lotteries, the small prizes do not form a strictly increasing sequence but are bunched together, with, say one set of $10 prizes, another of $50 prizes and so on. This prize structure is presumably adopted for simplicity. The precise structure of the set of small positive prizes does not make much difference to either the statistical expectation or the anticipated value of the lottery. These will be determined primarily by the probability of losing and by the value of the main prizes at the extreme upper tail of the distribution.

Second, the solution above is derived on the assumption that all lottery purchasers have identical preferences. The obvious market response to heterogeneous consumers is product differentiation. Consumers with limited risk-aversion and heavy overweighting of extreme low probability events will prefer lotteries dominated by one large prize while those with higher risk-aversion and more weight on events not quite so close to the tail will prefer a more even distribution of prizes. Obviously, a large group will simply abstain, including both global risk-averters and those whose anticipated utility gain from actuarially fair lotteries is insufficient to offset the operator's margin in an unfair lottery. If product differentiation is not possible, the solution would involve a compromise between different groups, with the weight depending both on numbers in different groups and the elasticity of demand with respect to deviations from the preferred prize structure.

Third, the analysis presented here assumes that the individual's income is known with certainty apart from a single lottery purchase. In practice there are numerous other sources of uncertainty and most people typically make a sequence of lottery purchases. As regards the first point, the upper tail of the wealth probability distribution will, for many people, be contributed primarily by the possibility of winning the lottery, since there is no prospect of attaining substantial wealth other than that associated with events, such as a random gift from a passing billionaire, which are even less probable than a win in the lottery. As regards the second, it is useful to consider the position of someone who is presented with a sequence of lotteries being resolved at regular intervals, for example, weekly, but whose time-horizon is longer, say a year. A somewhat more complex version of the analysis presented above is applicable, in which the individual's anticipated utility is determined by the probability distribution yielded by the sum of the different lotteries in which they participate. Thus the sum of the lotteries will yield a distribution similar to that of the optimal single lottery derived above. This implies that the optimal winning probability for any one lottery is smaller as is the optimal ticket price. The rational strategy in this case will be one in which a small number of tickets is purchased each week. The most obvious alternative would be one in which a large number of tickets were purchased in the first available lottery and no more subsequently. However, this strategy would be dynamically inconsistent, since unless a major prize was won, the position after the lottery was resolved would again be one in which it was rational to buy tickets.

Fourth, lotteries are only one of a number of gambling forms and others do not obviously fit the pattern predicted here. This reflects the fact that lotteries are a very pure form of gambling with no element of skill and little of the entertainment available from time-consuming forms of gambling such as bingo and poker machines. The analysis of lotteries presented here forms a useful basis for the treatment of more complex cases such as that of race-track betting. An analysis of this problem is presented as an indication of the way in which the insights derived form the simple model presented here may be applied to more complex cases.

VI. RACETRACK BETTING

Whereas the main interest in lotteries is in bets at very long odds, racetrack betting involves a wide range of odds from very short (odds-on) to very long (such as doubles and trebles). Clearly, a 'pure' gambling explanation of the kind advanced for lotteries will not work in this case and it is necessary to introduce other explanatory factors. The central distinguishing feature of racetrack betting is that there are no objectively given odds, and there is considerable room for disagreement about the prospects of particular horses. The existence of such disagreements would provide a reason for bets between risk-neutral or risk-averse individuals. On the other hand, the incentives for information collection and the well established markets for information provision mean that it is unlikely that the market odds for bets between risk neutral individuals could differ greatly from the true odds. Thus it would appear that an explanation based on disagreements over odds must be integrated with an analysis of risk preferences of the type presented for lotteries.

This argument may be elaborated by assuming there are two groups of bettors. Members of the first group are risk-neutral, and invest substantial effort in assessing the winning probabilities of different horses. Despite the efforts of this group, there remains some noise in the system, leading to divergent estimates of winning probabilities. Thus, members of this group would be willing to place at least some bets even under a totalisator or pari-mutuel system in which it was certain that bettors as a group would lose. Members of the second group ('mug punters') have RDEU preferences of the kind set out above and are attracted by long-odds bets, even when they are actuarially unfair. They have no information about the merits of the horses and choose among the long-priced horses on a frivolous basis, such as attractive names or lucky numbers. This group will normally have a lower elasticity of demand for bets than the first group, for whom a small reduction in odds is likely to render bets unacceptable. Hence a monopolistic bookmaker will choose to offer odds on long-priced horses which are highly unfair. This will ensure that the bets of the first group are concentrated on short-priced horses and, therefore, that the elasticity of demand for bets on these horses is high and the bookmaker's margin correspondingly low.

A number of writers including Griffiths (1949) and Ali (1977) (see Bird and McCrae 1984 for a summary of the literature) have examined the structure of returns to racetrack bets. These writers examine the average return per dollar invested on bets at particular odds or on horses in order of favoritism. Essentially the same features are found by all writers. First, all bets of this kind are actuarially unfair. This is not surprising, since otherwise there would exist a profitable betting strategy requiring no information about the individual characteristics of the horse involved. Second, the return to punters is near unity for short-priced horses, but falls as low as 60 cents in the dollar for long-priced horses. This is consistent with the previous discussion. Bets on long-priced horses are a substitute for lottery tickets, for people who know nothing about the prospects of the horses concerned. Thus, their rate of return should approach that of a lottery ticket as the odds increase, As a result of taxes and operating expenses, this latter return is typically of the order of 60 cents in the dollar. The general pattern of returns on ‘exotic’ bets such as exactas and daily doubles is similar (Asch and Quandt 1987).

This suggests that the returns structure may be regarded as being determined in the following fashion. As a first approximation, the return on a bet on a particular horse may be regarded as being determined by the objective odds and the amount people are willing to pay for a bet at those odds in the general gambling market. this would yield exactly fair returns for bets at evens with a gradually decreasing rate of return as odds lengthen. Because of disagreements regarding the appropriate odds for a particular horse, bookmakers are able to offer odds slightly less favorable than those suggested by this rule, and in particular to lay bets on short-priced horses which are, on average, slightly unfavorable to punters.

Thus, RDEU theory permits a fairly straightforward explanation of the demand side of the market for racetrack bets. There are, however, a number of interesting problems on the supply side. The fact that the margins differ substantially between the different bets suggests that bookmakers are able to exercise a degree of price discrimination, which in turn implies the exercise of market power. In the case of totalisator or pari-mutuel betting, this pattern of payments is generated automatically in response to bettor demand, but it is not clear precisely how the same result is achieved by on-course bookmakers, since the market appears to be fairly competitive. As with other discriminating monopoly solutions, the existing pattern is profitable for bookmakers as a group, but appears to offer incentives for any individual to defect. At first sight, it would appear profitable for an individual bookmaker to specialize in taking bets on long-priced horses, and offer slightly better odds than those who took bets on the field. A number of possible explanations may be advanced including the existence of informational difficulties or the operation of informal sanctions which act to preserve a cartel, but none appears completely convincing at this stage.

VII. CONCLUDING COMMENTS

Rank-Dependent Expected Utility theory is a generalization of the Expected Utility approach which is currently the basis of most economic analysis of behavior under uncertainty. In order to show that this increased generality is more than mere formalism, it is necessary to demonstrate that RDEU theory not only resolves anomalies which have arisen in EU analysis, but also that it permits the extension of economic analysis into new areas of behavior. Gambling is one of the most important aspects of economic behavior which has not until now been amenable to analysis. At least in the case of lotteries, RDEU theory appears to give a very satisfactory account of gambling behavior and gambling markets.

The potential scope of the theory is much wider. A wide range of investments, such as shares and futures contracts include low-probability high return events at the upper tail of the distribution of possible outcomes. It seems likely that use of an EU model with coefficients derived from standard questionnaire methods will lead to underestimates of the attractiveness of such investment. Conversely, EU methods may give an excessively favorable evaluation of private investments or public policy choices involving low probability events with very severe losses attached. Decisions associated with the siting of nuclear power plants provide an obvious example.

Observed behavior in the face of risk is dominated by risk aversion, but contains elements which seem to imply risk preference. Under EU theory, this seems contradictory or irrational. The RDEU approach permits the development of a portfolio analysis which will include elements of risk-seeking along with risk aversion.

APPENDIX

Proof of Proposition 1:

Two lemmas are required, before the main result is proved.

Lemma 1. Let p^* be the probability which satisfies the condition (18) for minimization of the weighted probability of loss, and let [i/N, (i+1)/N] be the interval of measure 1/N containing p^*. Let Z be any prize structure such that the probability, o(p,˜), of the lowest prize does not lie within this interval. Then there exists a preferred prize structure which does satisfy this condition.

Proof: Assume without loss of generality that the lowest prize is -1, and denote the next prize by z_1. Suppose o(p,˜) $= k/N$, $k<$□ Then $h_{k+1}(\boldsymbol{p})$□□$q(k/N)/(k/N)$.

Hence, $q(k/N)\, U(-1) + h_{k+1}(\boldsymbol{p})\, U(z_1) < (q(k/N) + h_{k+1}(\boldsymbol{p}))\, ((k\, U(-1) + U(z_1)/(k+1))$

$$< q((k+1)/N)\, U\,((-k+z_1)\,/\,(k+1)),$$

by the concavity of U.

Hence Z will be dominated by a lottery with lowest prize $(-k+z_1)\,/\,(k+1)$ arising with probability $(k+1)/N$. The converse case, where o(p,˜) $= k/N$, $k >$□$+1$, follows similarly with the observation that in a sufficiently small neighborhood of any point U is ‘close enough’ to linearity.

Lemma 2: Let Z be a lottery in which the probability of the minimum prize satisfies the optimality condition derived in Lemma 1, and let i be such that

(i) $w_i > w_{i-1}$

(ii) $w_{i+1} = w_i$.

then there exists a preferred lottery with $w_{i+1} > w_i$.

Proof: By the conditions on the minimum prize $q'(p)$ is increasing over the relevant range and

$h_{i+1} > h_i > h_{i-1}$.

Hence there exists some □ such that

(i) $w_i - \square > w_{i-1}$

(ii) $h_i\, U(w_i) + h_{i+1}\, U(w_{i+1}) < h_i\, U\,(w_i - \square) + h_{i+1}\, U(w_{i+1} + \square)$

If $w_{i+1} + \square < w_{i+2}$, then the preferred lottery may be constructed by reducing prize *i* by □, and increasing prize *i*+1 correspondingly. If this condition is not satisfied, then it is necessary to spread the increased prize money over a range of higher valued prizes to preserve the non-decreasing sequence of prizes, but a condition analogous to (ii) will still hold.

The main result may now be proved inductively. After the sequence of minimum prizes satisfying lemma 1, consider the two subsequent prizes in an optimal lottery design. By Lemma 2, they will form part of an increasing sequence. Lemma 2 may now be applied to each of the subsequent prizes in turn.

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