On the Limitations of Monetary Policy

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Abstract

This paper argues that in a homogeneous monetary Real Business Cycle economy where a complete set of nominal contingent claims exist, the requirement to collateralize loans, alone, does not affect the equilibrium allocation when monetary policy is chosen optimally: the Pareto optimal allocation can be supported. Rather, it is the presence of additional inefficiencies such as market incompleteness or heterogeneity of agents that limits the ability of optimal monetary policy, and more precisely, inflation, to support the first best allocation. In our model policy is non-Ricardian or equivalently outside money exists, and the Central Bank trades only in short-term nominally risk-free bonds: as a consequence monetary policy that sets rates of interest and accommodates money demand effectively determines the allocation of prices at equilibrium. A Friedman rule (r=0), which would be optimal in the absence of collateral constraints, here it is not: at the resulting prices collateral constraints bind. A path of prices that avoids binding collateral constraints necessarily involves a non-zero interest rate. The path of prices that supports the Pareto optimal allocation occurs when the collateral constraint binds: a positive inflation tax on money balances is efficient. For interest rates that permit the collateral constraint to bind, a policy of stable inflation (alternatively, money growth) implies that the collateral constraint binds after a sequence of positive (respectively negative) productivity shocks followed by a negative (respectively positive) productivity shock.

Key words: monetary policy; inflation; collateral.

JEL classification numbers: E13; E22; E41; E44; E52; E61
1 Introduction

In a monetary economy, under complete markets, the demand for money accommodates an optimal consumption-investment plan. Agents can tailor their state-contingent demand for money to satisfy their optimal plan for any given path of prices. In this paper the demand for money derives from a cash-in-advance constraint: holding money balances is costly under positive interest rates. As a consequence the optimal path of prices is usually supported by zero interest rates: the Friedman rule obtains. Note, however, that under such a policy there exists an open set of prices which support a unique real allocation: the indeterminacy is purely nominal.

We deviate from the complete markets paradigm by requiring short-sales of assets, state-contingent bonds in particular, to be backed by collateral. The value of collateral at any future state cannot be less than the face value of the bond maturing at that state. The possibility that the collateral constraint may bind means that optimal interest rates cannot be zero: the Friedman rule fails. If the constraint binds under such a policy, then the optimal decision to hold (real) capital will be determined jointly with the decision to hold assets. It follows that if neither money demand nor the path of prices can be determined then the real allocation also cannot be unique: the indeterminacy becomes real. Therefore the optimal interest rate policy must involve strictly positive interest rates if the constraint is to bind at zero interest rates. However this comes at a cost: positive interest rates, through the corresponding inflation cost incurred in holding money balances, reduce the efficiency of invested capital (one can think of this as reducing the total productivity of a given level of capital) and hence reduces the equilibrium rate of investment.

Monetary policy affects the rate of inflation and, thus, the future value of capital, which acts as collateral. As a result, in the presence of financial frictions, the rate of inflation determines the equilibrium allocation. In a monetary economy, the nominal cost of borrowing, i.e. the interest rate, and financial frictions interact. In such a setting there is a role for monetary policy to determine the equilibrium allocation, as presented in Tsomocos (2003) and Goodhart et al. (2006)\textsuperscript{1}. Here we show that, when a complete set of nominal

\begin{footnotesize}
\textsuperscript{1}In these models, the demand for money is supported by cash-in-advance constraints and financial frictions are explicitly introduced through endogenous default on nominal obligations. Shubik and Yao (1990), Shubik and Tsomocos (1992) and Shubik and Tsomocos (2002) present the importance of monetary transaction costs and nominal wealth
\end{footnotesize}
contingent claims are available, a collateral constraint (that excludes the possibility of default\textsuperscript{2}) cannot affect the allocation when monetary policy is chosen optimally: the Pareto optimal allocation can be supported. In order to analyse the trade-off between financial frictions, monetary policy and financial stability, market incompleteness and agent heterogeneity are an important modelling characteristic, as advocated in the aforementioned papers.

Collateral constraints reduce the ability of agents to borrow after a threshold. If the threshold is exogenous, this would reduce both the amount the individual agent can invest and consume. Under a collateral constraint borrowing can be increased if supported by additional capital investment. As this is preferred, capital investment increases while consumption is left sub-optimally low: the premium generated by a binding collateral constraint increases borrowing only to support investment. Monetary policy distorts this premium because agents cannot collateralize their money balance: money, as opposed to physical capital is easier to be hidden and so harder to be seized. This assumption is analogous to the role that non-collateralizable trees play in Kiyotaki and Moore (1997). Raising nominal interest rates increases the rate of inflation and erodes the real value of the money balances. As their real wealth falls in the future, agents reduce the amount they borrow and, via the collateral constraint, the amount of capital they invest. The optimal path of interest rates exploit the trade-off between the higher investment which a binding collateral constraint results in, with the lower level of investment which positive expected interest rates induce. If the constraint binds at positive interest rates, the optimal path of interest rates sets a rate of inflation to tax money balances such that the premia associated with the binding collateral constraint exactly offsets the future costs of inflation incurred by positive expected interest rates. That the initial level of output is pre-determined allows for the existence of such a solution. In other words, the optimal policy

\textsuperscript{2}The complete set of nominal contingent claims and the presence of a representative agent implies that default on collateralized loans cannot have any real effects. Collateral constraints that exclude default can result in an inefficient level of borrowing. Alvarez and Jermann (2000) within a real endowment economy with complete markets show that borrowing limits, to exclude default, can be chosen such that the constrained efficient allocation is attainable. We say that appropriate monetary policy can go a step further and achieve the first-best in a representative agent economy with production and restrictions on borrowing.

\textsuperscript{2}Goodhart et al. (2010) and Li et al. (2011) extend their framework to account for deflationary pressures on collateral.
function allows the planner to determine a level of investment in the presence of positive interest rates, as if interest rates were zero: the first best obtains.

Our model is a monetary Real Business Cycle model, where money is introduced along the lines of Lucas and Stokey (1987). There is a single homogeneous good which is produced each period that is allocated between consumption and investment for future production. We introduce into this framework a collateral constraint as is found in Kiyotaki and Moore (1997). Thus, agents can only borrow as much as the nominal value of the capital they provide. Finally we analyse this framework within a monetary complete markets Arrow-Debreu economy (see Nakajima and Polemarchakis (2005)), where a monetary authority is willing to purchase debt obligations from households, specifically a full set of state contingent bonds, in exchange for money, at a given interest rate. The monetary authority sets the price of the sum of the bond prices (the nominal risk free rate) while the households determine the price of individual securities.

We first examine the optimality of monetary policy in a deterministic economy, where the only choices for the monetary authority involve setting nominal interest rates. When we consider an economy with stochastic productivity shocks monetary policy involves a richer menu of choices. In this setting we consider two commonly used policy objectives concerning the path of prices. Namely, the monetary authority can choose either to target a stable growth in money supply (monetary stability), or a stable growth rate in prices (price stability). The choice of either regime implies when the collateral constraint will bind. Monetary stability implies that prices fluctuate inversely with productivity shocks. A positive productivity shock depresses the price level, and hence causes the collateral constraint to bind there. On the other hand, price stability is achieved by increasing the amount of aggregate credit (i.e. money) in states when aggregate production is low: the collateral constraint binds because the face value of debt is higher. Finally when we study an infinite horizon recursive economy we show that under a policy of price stability the collateral constraint binds after a sequence of positive productivity shocks followed by a negative one and is reminiscent of the Minsky (1992) financial instability hypothesis\(^3\). Under a policy of monetary stability a sequence of negative productivity shocks followed by a positive one results in the constraint binding.

\(^3\)Bhattacharya et al. (2011) show how the interaction of learning, risk-shifting, endogenous leverage and default can generate Minsky cycles.
Collateral constraints resolve an important issue which classical general equilibrium models abstract from: debtors have an incentive to default on their debt obligations. Bernanke and Gertler (1986) and Kiyotaki and Moore examine the issue by introducing a constraint on borrowing such that it cannot exceed in value some collateral pledged. As capital is central in the Real Business Cycle literature, it plays the role of collateral. What is central in such studies, is the point in time and state of the world where the value of collateral/capital is calculated. Models considering borrowing constraints that depend on the current or expected value of collateral deviate from the underlying principle of posting collateral, which is to protect lenders from debtors’ incentive not to repay. Herein, we condition current borrowing on the future (state-contingent) value of collateral following closely the contribution by Kiyotaki and Moore. We also deviate from their framework in a number of ways. First, agents in our economy do not always borrow to the limit such that the collateral constraint binds, but rather is an outcome of the monetary policy implemented. Second, in the absence of a binding constraint, our economy has full span. An attractive feature in Kiyotaki and Moore is that the combination of binding collateral constraints and of a negative productivity shock result in a cycle of falling capital, hence collateral, prices and lower borrowing. Krishnamurthy (2003) within a Kiyotaki-Moore economy shows that hedging against the event of binding constraints neutralizes the spiral of falling collateral prices and lower borrowing. In order to hedge against every eventuality his model allows for a complete set of Arrow securities as we do herein. Nevertheless, demand of such securities is unrestricted in Krishnamurthy. This is our third deviation from the mainstream framework as we present a monetary economy, in which Arrow securities are supplied by the monetary authority satisfying portfolio constraints. The demand for Arrow securities will depend on the interest rate target this authority sets. Hence, the value of collateral pledged does not only depend on productivity shocks, but also on monetary policy to set a path of prices. Consequently, we also derive cases under which the collateral constraint binds under positive productivity shocks. But, most importantly we show how investment decisions vary across realizations of binding and non-binding collateral constraints. The monetary objective to set and determine prices interacts with collateral constraints. In particular, we show that a determinant optimal path of prices should involve binding constraints, contrary to what Krishnamurthy would have derived as his focus is on a real economy.

The collateral constraint we assume is not unique. An alternative ap-
approach would have been to allow agents to choose an endogenously determined loan-to-value ratio with the possibility of borrowers’ default, should the price of collateral be less than the promised repayment. Geanakoplos and Zame (2002) consider a real economy where default on collateralized loans can be optimal. However, Geanakoplos (2003) and Fostel and Geanakoplos (2008) show that a collateral constraint, which excludes default is indeed optimal and will be chosen by atomistic agents under certain assumptions, such as two states states of future uncertainty as in our paper. Nominal contracts and monetary policy are not considered. Goodhart et al. (2010) and Li et al. (2011) examine default in collateralized contracts within a monetary economy. The former consider an incomplete markets economy with a banking sector and collateralized nominal loans, and show that monetary policy affects both the level of default, hence financial stability, and the real allocation. The latter examine an additional real effect of default on nominal contracts, which is the (costly) reallocation of productive capital. Essentially, money not repaid on collateralized loans will end up to other loan markets raising the cost of borrowing when the monetary authority fixes the money supply. This distorts investment decisions, since capital does not accrue to the hands of agents with the higher marginal productivity. This channel affects optimal production due to the heterogeneity between constrained and unconstrained agents, which this paper abstracts from. In this paper we consider an economy with a complete set of nominal contingent claims and show that there is no overall inefficiency in a monetary economy with a collateral constraint when monetary policy is chosen appropriately. As the aforementioned papers suggest, optimal policies may not exist when a complete set of nominal contingent claims do not exist, or equivalently, when default is not ruled out. This provides additional roles for fiscal policy or financial regulation, as in Goodhart et al. (2011).

The rest of the paper proceeds as follows. We first present the benchmark two-period deterministic economy in Section 2 where the channel through which monetary policy works, and the optimality of policy is evaluated. Section 3 extends the benchmark economy to include stochastic productivity shocks in the second period. We obtain implications for policy objectives of Price and Monetary Stability as well as derive optimal monetary policy/interest rates. Section 4 characterises an infinite horizon/recursive economy, obtaining the properties of the two period economies, the implications of policy objectives following a sequence of productivity shocks and provides the optimal policy function. Finally, 5 provides some final remarks about the
implications and limitations of the results.

2 Benchmark Economy

2.1 A Model without Uncertainty

Nakajima and Polemarchakis (2005) consider an economy with a representative agent smoothing his consumption over time and being subject to cash-in-advance constraints, as in Lucas and Stokey (1987). All markets are competitive and prices are flexible. We deviate from Lucas and Stokey by considering a non-Ricardian monetary policy, i.e. outside money exists, such as in Woodford (1996), Tsomocos (2003), Dubey and Geanakoplos (2006), and Espinoza et al. (2009). Monetary policy is conducted through trading nominal bonds to achieve predetermined interest rate targets. We introduce a durable capital used for production and require that bonds sales by households are backed by collateral. Bonds purchases by the monetary authority are guaranteed when the value of collateral is higher than the repayment on the bond in any state of the world. Here we describe a simple two period model without uncertainty to show the key results of the paper. In subsequent sections we generalise the benchmark economy to include stochastic aggregate productivity shocks, then fully characterise a recursive economy. The structure of the benchmark model is otherwise identical to subsequent sections.

2.2 Households

There are three periods: \( t = \{0, 1, 2\} \). Production and consumption occur in the first two periods. The last period is added for an accounting purpose, where households and the fiscal authority redeem their debt.

There is a continuum of identical households, distributed uniformly over \([0, 1]\). At each date-event, households produce a single, homogeneous product. The output produced by a representative household is \( y(0) \) and \( y(1) \), in period 0 and period 1, respectively. Similarly, consumption is denoted by \( c(0) \) and \( c(1) \).

Agents are endowed with \( y(0) \) of output in period 0. Agents consume some proportion, \( c(0) \) of this and invest \( k(1) \) for use as capital in the next period. Output in period 1 is \( y(1) = y(k(1)) \), i.e. capital is the only factor
of production. We use a decreasing returns to scale production function for output, \( y(1) = A(1)k(1)^\alpha \).

The preferences of the representative household are described by a linear lifetime expected utility:

\[
c(0) + \beta c(1).
\]

The representative household enters the initial period 0 with nominal assets \( w(0) \). At the beginning of the period, the asset market opens, in which cash and bonds are traded. Let \( r(0) \) be the nominal interest rate in period 0, thus \( \frac{1}{1+r(0)} \) be the price of a bond that pays off one unit of currency in the next period. The market for goods opens next. The purchase of consumption goods is subject to the cash-in-advance constraint. The period 0 budget constraint is then

\[
m(0) + p(0)[c(0) + k(1)] \leq w(0) + \frac{b(1)}{1 + r(0)} + p(0)y(0),
\]

where \( k(1) \) is the capital investment and \( b(1) \) the portfolio of bonds. The transactions of the household in the second period are similar. The nominal interest rate in period 1 is \( r(1) \). The flow budget constraint that the household faces is

\[
m(1) + p(1)c(1) + b(1) \leq m(0) + \frac{b(2)}{1 + r(1)} + p(1)y(1),
\]

at the end of the second period, all debt is repaid

\[
b(2) \leq m(1).
\]

Concerning the timing of transactions we assume that at each date-event the asset market opens before the goods market. An important consequence of this assumption is that the cash the households obtain from sales of its output has to be carried over to the next period. This is equivalent to householders carrying cash balances equivalent to the receipts of sales. Formally, the cash-in-advance constraints are

\[
m(0) \geq p(0)y(0),
\]

\footnote{This is chosen to simplify calculations but has the additional benefit that the results do not depend on concavity of the utility function.}
and

\[ m(1) \geq p(1)y(1). \] (6)

The collateral constraint limits the quantity of state contingent bonds the householder can short sell in period 0 to the nominal value of its capital, which serves also as collateral, in period 1. Formally\(^5\):

\[ b(1) \leq p(1)k(1). \] (7)

When the collateral constraint does not bind, the first order equation for the bond is

\[ \frac{1}{1 + r(0)} = \beta \frac{p(0)}{p(1)}. \] (8)

Similarly, the first order equation for capital is

\[ k(1)^{1-\alpha} = \alpha \beta A \frac{1}{1 + r(1)}. \] (9)

Given that we consider a risk-neutral agent we will need to make further assumptions on productivities to restrict the equilibria under consideration to be interior ones. As the production function is a continuous concave function of capital, an interior solution always exists given choices of \( y(0), A(1), \alpha, \beta, r(1) \) satisfying

**Assumption 1.** \( \frac{\alpha \beta A(1)}{1 + r(1)} \frac{1}{1-\alpha} < y(0), \)

which ensures that the total amount of capital invested must be less than the date 0 endowment.

**Assumption 2.** \( \frac{\alpha \beta}{1 + r(1)} < 1, \)

which ensures that \( k(1) < A(1)k(1)^{\alpha} = y(1). \) That is, that the production technology is profitable.

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\(^5\)We examine other specifications of the collateral constraint in the Appendix and show that our key results do not depend on it.
2.2.1 Monetary Authority

The Monetary authority budget constraints are similarly defined. The monetary authority sets exogenously targets the level of interest rates in each period, and subsequently acts in the bond market to accommodate the demand for money by households at the pre-specified interest rates. When the authority purchases bonds sold by agents, it injects money into the economy. At maturity, households repay their debt with money and the monetary authority has to satisfy its budget constraints. In period 0, the money supplied to the economy, \( M(0) \), is equal to total assets of the monetary authority, which are the portfolio of bonds it has purchased plus the initial monetary wealth \( W(0) \). Hence, for period 0:

\[
M(0) = \frac{B(1)}{1 + r(0)} + W(0). \tag{10}
\]

In period 1, the change in the money supply, \( M(1) - M(0) \), is funded by the change in value of the portfolio of bonds the authority holds, \( \frac{B(2)}{1 + r(1)} - B(1) \), given that households repay their debt obligations in full. Thus, the budget constraint of the monetary authority is period 1 is:

\[
M(1) + B(1) = M(0) + \frac{B(2)}{1 + r(1)}. \tag{11}
\]

Finally, in Period 2, the monetary authority receives repayment on its asset holding, \( B(2) \), and cancels its existing liability, \( M(1) \), i.e.

\[
M(1) = B(2). \tag{12}
\]

The present-value budget constraint gives:

\[
M(0) \frac{r(0)}{1 + r(0)} + M(1) \frac{r(1)}{1 + r(0)1 + r(1)} = W(0) \tag{13}
\]

where \( M(0) \) and \( M(1) \) are money supplies, \( W(0) \), \( B(1) \), \( B(2) \) are the total liabilities of the monetary authority. The second period budget constraint of the monetary authority can be combined with the third period one to give

\[
B(1) = M(0) - \frac{r(1)}{1 + r(1)}M(1) \tag{14}
\]

Monetary policy sets nominal interest rates, \( r(0) \geq 0 \) and \( r(1) \geq 0 \).
2.3 Equilibrium conditions

Since households are identical, individual consumption plus investment are equal to individual production. Moreover, the money stock is used for the purchase of produced goods and the money balance of the representative household at the end of each period is equal to the total money supply. Finally, bond sales by households are equal to bond purchases by the monetary authority. Thus, the market clearing conditions are

\[
\begin{align*}
  c(0) + k(1) &= y(0), \quad c(1) = y(1), \\
  m(0) &= M(0), \quad m(1) = M(1), \\
  b(1) &= B(1), \quad b(2) = B(2).
\end{align*}
\]

Also, consistency requires that

\[ w(0) = W(0). \]

A competitive equilibrium with interest rate policy is defined as follows:

**Definition 1.** Given initial nominal wealth, \( w(0) = W(0) \), interest rate policy, \( \{r(0), r(1)\} \), a competitive equilibrium consists of an allocation, \( \{c(0), c(1)\} \), a portfolio of households, \( \{m(0), m(1)\} \), a portfolio of the monetary-fiscal authority, \( \{M(0), M(1), B(1), B(2)\} \), spot-market prices, \( \{p(0), p(1)\} \), such that

1. the monetary authority accommodates the money demand, \( M(0) = m(0) \) and \( M(1) = m(1) \);
2. given interest rates, \( r(0), r(1) \), spot-market prices, \( p(0), p(1) \), the household’s problem is solved by \( c(0), c(1), k(1), m(0), m(1), b(1), \) and \( b(2) \);
3. all markets clear.

2.4 Determinacy and the Non-Neutrality of Monetary Policy

The cash-in-advance constraints result in a positive demand for money and positive interest rates are a necessary condition such that the model exhibits nominal determinacy. However, this comes with a loss in efficiency even in
the simple two period model under consideration. Consider a case where agents are unconstrained in their investment decisions. Their utility with respect to the investment decision, \( k(1) \), after imposing market clearing is

\[
y(0) - k(1) + \beta y(1) = y(0) - k(1) + \beta A(1)k(1)^\alpha
\]

Thus, the optimal investment in capital, which maximizes welfare, is

\[
k(1)^{1-\alpha} = \alpha \beta A(1)
\]

Nevertheless, capital investment under positive interest rates in the competitive economy described above is

\[
k(1)^{1-\alpha} = \frac{\alpha \beta A(1)}{1 + r(1)}.
\]

Although positive interest rates are necessary for a determinate equilibrium, they also result in lower capital investment, \( k(1) \) than the optimal one. The monetary authority may try to keep interest rates as low as possible, as in the Friedman rule where they approach zero. We show in the following section that this ceases to be optimal in an economy with collateral constraints. In particular, the limiting case with interest rates approaching zero can result in binding constraints for an open set of economies, which results in an inefficient level of investment. One of the objectives of the paper is to examine the ability of monetary policy to achieve the optimal investment allocation.

At this point it is important to note that, in general, in a finite horizon monetary economy, interest rates need to be positive only at terminal nodes. Let us assume that \( r(0) = 0 \) and \( r(1) > 0 \). In this case, all the seigniorage will be returned in the second period and hence determining \( M(1) \) and \( p(1) \). \( p(0) \) can then be determined from the Fisher equation, using the equilibrium values of the state price. However, \( M(0) \) is indeterminate as the cash-in-advance constraint will not bind.

Assume now an economy with stochastic productivity shocks, as in Section 3, in which \( r(0) > 0 \) but \( \forall s \in S, r_1(s) = 0 \). In this case, all the seigniorage will be returned in the first period and as the cash-in-advance constraint will not bind in the second, the money supplies there will be indeterminate. The no-arbitrage condition now only determines the average rate of inflation, but the distribution of inflation is again indeterminate. In
other words, with non-vanishing nominal rates of interest, it is unnecessary to tax away the entire amount of balances at the initial date, as such a value is progressively eroded by subsequent positive rates of interest.

In order to illustrate the complexity that binding collateral constraints introduce, observe that for an economy with stochastic productivity shocks, as in Section 3, the optimality condition for capital investment, when the constraint binds in state \( s \), becomes

\[
k(1)^{1-\alpha} = \alpha \beta \sum_s \pi(s) \frac{A_1(s)}{1 + r_1(s)} + \lambda_s p_1(s) k(1)^{1-\alpha},
\]

where \( \lambda_s \) is the Lagrange multiplier associated with the binding collateral constraint \( b_1(s) = p_1(s) k(1) \).

As discussed, the price level is indeterminate in a stochastic economy when \( r_1(s) = 0 \) \( \forall s \in S \). Consequently, the capital investment \( k(1) \) satisfying the equation above cannot be determined as well, and nominal indeterminacy under the Friedman rule manifests itself as real indeterminacy. Thus, there is a necessity for the monetary authority to set positive interest rates to determine the allocation. However, this comes with a loss of efficiency in the accumulation of capital. In the benchmark deterministic economy that we consider here, we will arbitrarily assume that interest rates at date 1 are positive, though when we generalise the model and characterise a full recursive equilibrium this assumption will be part of the optimal monetary policy choice.

### 2.5 Equilibrium Analysis

Here we consider the allocation without and with a binding collateral constraint.

#### 2.5.1 Equilibrium Analysis without Binding Collateral Constraints

We can solve for the allocation as follows. Using (9) and market clearing

\[
y(0) = c(0) + k(1) = c(0) + \left\{ \alpha \beta A(1) \frac{1}{1 + r(1)} \right\}^{1/(1-\alpha)}
\]

Using the production function, we get

\[
c(0) = y(0) - \left\{ \alpha \beta A(1) \frac{1}{1 + r(1)} \right\}^{1/(1-\alpha)}.
\]
We can similarly solve for period 1 variables

\[ y(1) = c(1) = A \left\{ \alpha \beta A(1) \frac{1}{1 + r(1)} \right\}^{\alpha/(1 - \alpha)}. \]  

(16)

We will now state an additional assumption which will allow us to prove the following proposition:

**Assumption 3.** \( \frac{\alpha}{1 + r(1)} > A(1)^{\alpha - 1} + \frac{r(1)}{1 + r(1)} \),

**Proposition 1.** Given assumption 3, there exists a date zero interest rate at which the collateral constraint first binds.

Proposition 1 implies that a positive level of inflation is necessary to avoid the collateral constraint from binding. Note however, that the optimality of this allocation is unclear. In the following propositions we show that although positive date 1 interest rates reduces the level of investment, the presence of the collateral constraint at interest rates approaching zero implies that the level of investment in equilibrium will be inefficiently high or equivalently, that the rate of inflation is inefficiently low.

### 2.5.2 Equilibrium Analysis with Binding Collateral Constraints

In this section we examine the implications for efficiency and policy when the constraint binds. That is, \( b(1) = p(1)k(1) \).

The period 0 budget constraint is then

\[ p(0)[c(0) + k(1)] \leq w(0) + \frac{p(1)k(1)}{1 + r(0)}, \]  

(17)

The transactions of the household in the second period are

\[ p(1)c(1) + p(1)k(1) \leq p(0)y(0) + \frac{p(1)y(1)}{1 + r(1)}, \]  

(18)

The first order conditions give us the following

\[ 1 - \alpha k(1)^{\alpha - 1} \beta A \frac{1}{1 + r(1)} = \left\{ \frac{1}{1 + r(0)} \frac{p(1)}{p(0)} - \beta \right\}, \]  

(19)

The liquidity premium that a binding collateral constraint produces is \( \frac{1}{1 + r(0)} \frac{p(1)}{p(0)} - \beta \): the rate of inflation is no longer the difference between the
nominal and real rates of interest. In the following sections we show that optimal monetary policy manipulates this premium to obtain the first-best (Pareto optimal) allocation. We can determine the rate of inflation from the binding collateral constraint \( b(1) = p(1)k(1) \). From market clearing, \( B(1) = b(1) \), and from the monetary authority budget constraint \( B(1) = M(0) - \frac{r(1)}{1+r(1)}M(1) \). From the cash-in-advance constraints \( p(0) = M(1)y(0) \) and \( p(1) = M(1)y(1) \). Using these results we get that, when the collateral constraint binds

\[
\frac{p(1)}{p(0)} = \frac{y(0)}{y(1)} \frac{k(1)}{g(1)} \frac{1}{1+r(1)} + \frac{r(1)}{1+r(1)} y(1) - \beta.
\]

2.6 Optimal Monetary Policy

In this section we characterise the optimal monetary policy. Here we characterise how investment responds to changes in the date 0 interest rate when the collateral constraint binds. Substituting equation 20, equation 19 becomes:

\[
1 - \alpha k(1)^{\alpha - 1} \beta A \frac{1}{1+r(1)} = \left\{ \frac{1}{1+r(0)} \frac{y(0)}{k(1)} + \frac{r(1)}{1+r(1)} y(1) - \beta \right\}
\]

Total differentiation with respect to \( r(0) \) yields that \( \frac{\partial k(1)}{\partial r(0)} < 0 \). That is, raising the date zero interest rate lowers the rate of capital accumulation. As we have shown in the previous section that the constraint first binds at a positive interest rate, this result implies that the level of investment is higher when the constraint binds than when it does not.

Now we obtain the optimal interest rate at date zero.

Proposition 2. Given date one interest rates, the optimal date zero rate is strictly less than that which relaxes the collateral constraint.

2.7 Discussion of Results

The collateral constraint behaves, very loosely, in much the same way as an exogenous restriction on the ability to borrow after a threshold. Usually this would reduce both the amount the individual agent can invest and consume. The collateral constraint allows the individual the opportunity to increase borrowing, if incremental debt is supported by additional capital investment. As this is strictly preferred by the agent, in equilibrium, the
agent increases his capital investment and reduces his consumption, compared to a situation where the collateral constraint does not operate. That monetary policy distorts the premium generated by the collateral constraint stems from the inability of agents to collateralize their total wealth carried forward: money balances cannot be collateralized. Raising the nominal interest rate at date 0 increases the cost of carrying money balances, and the agent can offset this effect by borrowing less and hence reducing the capital investment. The optimality of monetary policy stems from the ability to exploit the trade-off between the higher investment which a binding collateral constraint results in with the lower level of investment which positive date 1 interest rates induce. We can show this most cleanly by considering an economy where we index the payoff of the bond to be $\frac{p(1)}{p(0)}$. Assume we are at an optimum where the collateral constraint just binds. The nominal value of debt borrowed at date 0 is now $\frac{p(1)}{p(0)}b(1)$. The Fisher equation still gives us that $\frac{1}{1+r(0)} = \beta \frac{p(0)}{p(1)}$, which means that the nominal value of debt borrowed at date 0 is $\beta b(1)$. The date 0 budget constraint can now be normalised to be $c(0) + k(1) = \frac{\beta b(1)}{p(0)} + \frac{w(0)}{p(0)}$. The date 1 budget constraint, combined with the date 2 budget constraint and the cash-in-advance constraints, becomes $c(1) + \frac{b(1)}{p(0)} = \frac{p(0)}{p(1)}y(0) + \frac{y(1)}{1+r(1)}$. Now raising the nominal interest rate increases the rate of inflation, reduces the real value of the money balances ($\frac{p(0)}{p(1)}y(0)$) at date 1. In addition, it can be shown that the date 0 price level falls, increasing the real value of debt to be repaid at date 1, while increasing the real value of income at date 0 ($\frac{\beta b(1)}{p(0)} + \frac{w(0)}{p(0)}$). As a result, an optimising agent can improve by reducing the quantity of bonds sold: the real value of debt repaid ($\frac{b(1)}{p(0)}$) must fall. When the economy is subject to a collateral constraint, this effect leads simultaneously to a reduction in the level of investment. To see this, note that the collateral constraint says $\frac{p(1)}{p(0)}b(1) = p(1)k(1)$. Normalising by $p(1)$ gives us $\frac{k(1)}{p(0)} = k(1)$. Hence if it is optimal to reduce $\frac{k(1)}{p(0)}$, then it must also be optimal to reduce $k(1)$. As the collateral constraint just binds, the overall effect will be that the level of investment will be unchanged in equilibrium, however the effects traced out here describe the mechanism through which inflation can affect the level of investment when the constraint is not at the margin.
3 Stochastic Economy

Here we generalise the results of the benchmark economy by considering stochastic aggregate productivity shocks.

3.1 Households

There are three periods: \( t = \{0,1,2\} \). A stochastic productivity shock, \( s \in S = \{1,\ldots,S\} \), realizes at the beginning of the second period. Each state occurs with a probability \( \pi(s) > 0 \). Production and consumption occur in the first two periods. The last period is added for an accounting purpose, where households and the fiscal authority redeem their debt.

There is a continuum of identical households, distributed uniformly over \([0,1]\). At each date-event, households produce a single, homogeneous product. The output produced by a representative household is \( y(0) \) and \( y_1(s) \), in period 0 and in state \( s \) in period 1, respectively. Similarly, consumption is denoted by \( c(0) \) and \( c_1(s) \).

Agents are endowed with \( y(0) \) of output in period 0. Agents consume some proportion, \( c(0) \) of this and invest \( k(1) \) for use as capital in the next period. At \( t = 1 \), \( s \) possible production shocks and interest rates can realize. Output in period 1 is \( y_1(s) = y(k(1)) \), i.e. capital is the only factor of production. We use a decreasing returns to scale production function for output, \( y_1(s) = A_1(s)k(1)^\alpha \).

The preferences of the representative household are described by a linear lifetime expected utility:

\[
c(0) + \beta \sum_s \pi(s)c_1(s). \tag{22}
\]

The cash-in-advance constraint at date 0 is

\[
m(0) \geq p(0)y(0). \tag{23}
\]

The period 0 budget constraint is then

\[
m(0) + p(0)[c(0) + k(1)] \leq w(0) + \sum_s q(s)b_1(s) + p(0)y(0), \tag{24}
\]

\(^6\)This is chosen to simplify calculations but has the additional benefit that the results do not depend on concavity of the utility function.
where \( k(1) \) is the capital investment and \( b_1(s) \) the portfolio of elementary securities.

Let \( r(0) \) be the nominal interest rate in period 0, and thus, \( 1/(1 + r(0)) \) be the price of a nominally riskless bond that pays one unit in every state of nature in the next period. The no-arbitrage condition then implies that

\[
\sum_s q(s) = \frac{1}{1 + r(0)}.
\] (25)

The transactions of the household in the second period are similar, except that it faces no uncertainty. The nominal interest rate in state \( s \in S \) is \( r_1(s) \). The flow budget constraint and the cash constraint that the household faces at state \( s \) are

\[
m_1(s) + p_1(s)c_1(s) + b_1(s) \leq m(0) + \frac{b_2(s)}{1 + r_1(s)} + p_1(s)y_1(s),
\] (26)

subject to the cash in advance constraint

\[
m_1(s) \geq p_1(s)y_1(s).
\] (27)

In the following period, the only economic activity the household conducts is the repayment of its debt:

\[
b_2(s) \leq m_1(s).
\] (28)

These combine for the lifetime budget constraint

\[
p(0)[c(0) + k(1)] + \sum_s q(s)p_1(s)c_1(s) \leq \frac{p(0)y(0)}{1 + r(0)} + \sum_s q(s)\frac{p_1(s)y_1(s)}{1 + r_1(s)} + w(0).
\] (29)

The collateral constraint limits the quantity of state contingent bonds the householder can short sell in period 0 to the nominal value of its capital, which serves also as collateral, in each state. Formally\(^7\):

\[
b_1(s) \leq p_1(s)k(1)
\] (30)

\(^7\)We examine other specifications of the collateral constraint in the Appendix and show that our key results do not depend on it.
When the collateral constraint does not bind the first order equations with respect to the state-contingent bonds gives

\[ q(s) = \beta \frac{\pi(s)p(0)}{p_1(s)}. \]  

(31)

The first-order condition for the optimal investment of capital, assuming the collateral constraint does not bind, simplifies to

\[ k(1)^{1-\alpha} = \alpha \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)}. \]  

(32)

Given that we consider a risk-neutral agent we will need to make further assumptions on productivities to restrict the equilibria under consideration to be interior ones. As the production function is a continuous concave function of capital, an interior solution always exists given choices of \( y(0), A_1(s), \alpha, \beta, r_1(s) \) satisfying

**Assumption 4.** \( A_1(L) = A_1(1) < A_1(2) < ... < A_1(S) = A_1(H) \).

**Assumption 5.** \( \{\alpha \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)}\}^{1-\alpha} < y(0), \)

which ensures that the total amount of capital invested must be less than the date 0 endowment.

**Assumption 6.** \( \{\alpha \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)}\} < A_1(L) \),

which ensures that \( k(1) < A_1(L)k(1)^\alpha = y_1(L) \). That is, that the production technology is profitable.

### 3.2 Monetary Authority

The Monetary authority budget constraints are similarly defined. Hence,

\[ M(0) = \sum_s q(s)B_1(s) + W(0) \]  

(33)

In period 1, state \( s \), the change in the money supply, \( M_1(s) - M(0) \), is funded by the change in value of the portfolio of bonds the authority holds,
Thus, the budget constraint of the monetary authority is state $s$ is:

$$M_1(s) + B_1(s) = M(0) + \frac{B_2(s)}{1 + r_1(s)}$$  \hfill (34)

Finally, in Period 2, given that state $s$ has previously realized, the monetary authority receives repayment on its asset holding, $B_2(s)$, and cancels its existing liability, $M_1(s)$, i.e.

$$M_1(s) = B_2(s).$$  \hfill (35)

The present-value budget constraint gives:

$$M(0)\frac{r(0)}{1 + r(0)} + \sum_s q(s)M_1(s)\frac{r_1(s)}{1 + r_1(s)} = W(0)$$  \hfill (36)

### 3.2.1 Determinacy and Monetary Authority Portfolio Choices

In order to ensure determinacy, we require the additional constraint on the relative quantities of state contingent bonds purchased by the Monetary authority. The reason is straightforward. As the equilibrium prices of the bonds are determined solely by the demands of the households, altering the relative quantities of bonds purchased by the monetary authority will then change the relative prices of the Arrow securities which leads to nominal indeterminacy under an interest rate rule (see Nakajima and Polemarchakis (2005)).

However, the allocation is determined given that markets are complete and that the collateral constraint does not bind. Define as $B_1$ the gross value of bonds purchased by the monetary authority in period 0. Also, denote by $f(s)$ the weight allocated to the state-contingent bond paying out in state $s$, where $\sum_s f(s) = 1$. Then,

$$B_1(s) = B_1 f(s)$$  \hfill (37)

and

$$B_1 f(s) = M(0) - \frac{r_1(s)}{1 + r_1(s)}M_1(s)$$  \hfill (38)
The monetary authority can select \( s - 1 \) portfolio weights, \( f(s) \), to fully determine the path of prices. Essentially, it can accommodate the money demand of households for a given real allocation determined by the predetermined interest rates, \( r(0) \geq 0 \) and \( r_1(s) \geq 0 \) (recall the optimality condition for capital investment). Nevertheless, the presence of the collateral constraint makes the demand of bonds depend on the capital investment decisions, since \( B_1 f(s) \leq p_1(s) k(1) \). Hence, the monetary authority cannot independently select its portfolio of bonds to pick a path of prices without affecting the level of real investment.

### 3.3 Equilibrium conditions

Since households are identical, individual consumption plus investment are equal to individual production. Moreover, the money stock is used for the purchase of produced goods and the money balance of the representative household at the end of each period is equal to the total money supply. Finally, bond sales by households are equal to bond purchases by the monetary authority. Thus, the market clearing conditions are

\[
\begin{align*}
c(0) + k(1) &= y(0), \\
c_1(s) &= y_1(s), \\
m(0) &= M(0), \\
m_1(s) &= M_1(s), \\
b_1(s) &= B_1(s), \\
b_2(s) &= B_2(s).
\end{align*}
\]

Also, consistency requires that

\[
w(0) = W(0).
\]

The no-arbitrage condition (25) implies that the prices of elementary securities, \( q(s) \), \( s \in \mathcal{S} \), can be written as

\[
q(s) = \frac{\mu(s)}{1 + r(0)}, \tag{39}
\]

for some \( \mu(s) \), \( s \in \mathcal{S} \), satisfying

\[
\sum_s \mu(s) = 1.
\]

It follows that \( \mu \) is viewed as a probability measure over \( \mathcal{S} \), and called the *nominal equivalent martingale measure*. A competitive equilibrium with interest rate policy is defined as follows:
**Definition 2.** Given initial nominal wealth, \( w(0) = W(0) \), interest rate policy, \( \{ r(0), r_1(s) \} \), a competitive equilibrium consists of an allocation, \( \{ c(0), c_1(s) \} \), a portfolio of households, \( \{ m(0), m_1(s) \} \), a portfolio of the monetary-fiscal authority, \( \{ M(0), M_1(s), B_1(s), B_2(s) \} \), portfolio weights of the monetary-fiscal authority \( f(s) \), spot-market prices, \( \{ p(0), p_1(s) \} \) and a nominal equivalent martingale measure, \( \mu \), such that

1. the monetary authority accommodates the money demand, \( M(0) = m(0) \) and \( M_1(s) = m_1(s) \);
2. given interest rates, \( r(0), r_1(s) \), spot-market prices, \( p(0), p_1 \), nominal equivalent martingale measure, \( \mu \), the household’s problem is solved by \( c(0), c_1(s), k(1), m(0), m_1(s), b_1(s), \) and \( b_2(s) \);
3. all markets clear.

### 3.4 Equilibrium Analysis

We will now characterise the properties of the economy with and without a binding collateral constraint.

#### 3.4.1 Equilibrium Analysis without Binding Collateral Constraints

In this section, we present the mechanics of the model and illustrate how the allocation can be solved for the case that the collateral constraint does not bind. Using (32) and market clearing

\[
y(0) = c(0) + k(1) = c(0) + \left\{ \alpha \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right\}^{1/(1-\alpha)}
\]

Using the production function, we get

\[
c(0) = y(0) - \left\{ \alpha \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right\}^{1/(1-\alpha)}
\]

We can similarly solve for period 1 variables

\[
y_1(s) = c_1(s) = A_1(s) \left\{ \alpha \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right\}^{\alpha/(1-\alpha)}
\]
The monetary sector can be solved as follows.

The lifetime budget constraint of the monetary authority is

\[ M(0) \frac{r(0)}{1 + r(0)} + \sum_s q(s) M_1(s) \frac{r_1(s)}{1 + r_1(s)} = W(0). \]  

(42)

Using the equilibrium value of the Arrow price

\[ M(0) = \frac{W(0)}{\frac{r(0)}{1 + r(0)} + \sum_s \frac{r_1(s)}{1 + r_1(s)} \pi(s) \frac{y_1(s)}{y(0)}}. \]  

(43)

The second period money supplies depend on the portfolio weights chosen. Choose any two states \( s', s^* \in S \)

\[ f(s') \frac{M(0) - M_1(s')}{f(s^*)} = \frac{M(0) - M_1(s)}{f(s^*)} \frac{r_1(s)}{1 + r_1(s^*)}, \]  

(44)

from which

\[ M_1(s') = \frac{1 + r_1(s')}{r_1(s')} \left[ 1 - \frac{f(s')}{f(s^*)} M(0) + \frac{f(s')}{f(s^*)} M_1(s^*) \frac{r_1(s^*)}{1 + r_1(s^*)} \right]. \]

The bond prices give us

\[ \sum_s q(s) = \sum_s \beta \pi(s) \frac{M(0)}{M_1(s)} \frac{y_1(s)}{y(0)} \]

\[ = \frac{1}{1 + r(0)} \]

From these equations, money supplies and prices can be solved for given values of \( f(s) \). In this paper we consider two possible choices of these parameters, determined by policy objectives of either price stability or monetary stability.

**Definition 3.** Price stability is the outcome of monetary policy that sets interest rates and prices in the second period which are state independent.

Formally, \( r(0), r_1(s) \geq 0 \) and a choice of \( f(s) \) such that \( \forall s \in S, p_1(s) = p_1 \).
Using this definition we can solve for equilibrium prices and money supplies as follows.

Take the no-arbitrage relationship

\[
\sum_s q(s) = \sum_s \beta \pi(s) \frac{p(0)}{p_1(s)}
\]

\[
\frac{1}{1 + r(0)} = \beta \frac{p(0)}{p_1}
\]

\[
p_1 = \beta \frac{M(0)}{y(0)} (1 + r(0))
\]

\[
= \frac{W(0) \beta}{y(0)} (1 + r(0))
\]

\[
= \frac{r(0)}{1 + r(0)} + \sum_s \frac{r_1(s)}{1 + r_1(s)} \pi(s) \frac{y_1(s)}{y(0)}.
\]

Using the quantity theory of money and the equilibrium value of capital invested we get \( M_1(s) = p_1 y_1(s) \). These can then be substituted into 44 to determine \( f(s) \).

**Definition 4.** Monetary stability is the outcome of monetary policy that sets interest rates and money supplies in the second period which are *state independent*. Formally, \( r(0), r_1(s) \geq 0 \) and a choice of \( f(s) \) such that \( \forall s \in S, M_1(s) = M_1 \).

Take the no-arbitrage relationship

\[
\sum_s q(s) = \sum_s \beta \pi(s) \frac{M(0)}{M_1(s)} \frac{y_1(s)}{M_1(s)} \frac{y_1(s)}{y(0)}
\]

\[
\frac{1}{1 + r(0)} = \beta \frac{M(0)}{M_1} \frac{y_1(s)}{y(0)} \sum_s \pi(s) \frac{y_1(s)}{y(0)}
\]

\[
M_1 = \beta M(0) \frac{y_1(s)}{y(0)} \frac{y_1(s)}{y(0)} (1 + r(0))
\]

\[
= \frac{W(0) \beta}{y(0)} \sum_s \pi(s) \frac{y_1(s)}{y(0)} (1 + r(0))
\]

\[
= \frac{r(0)}{1 + r(0)} + \sum_s \frac{r_1(s)}{1 + r_1(s)} \pi(s) \frac{y_1(s)}{y(0)}.
\]

Using the quantity theory of money and the equilibrium value of capital invested we get \( p_1(s) = \frac{M_1}{y_1(s)} \). These can then be substituted into 44 to determine \( f(s) \).
3.4.2 Equilibrium Analysis, Monetary Policy and the Collateral Constraint

In this section we consider different policy regimes and their ability to ensure that the collateral constraint does not bind in equilibrium. We first characterise the collateral constraint fully; then examine the implications for equilibrium when it does bind.

The household budget constraint and market clearing in period 1 gives us that

\[ b_1 f(s) = p(0)y(0) - \frac{r_1(s)}{1 + r_1(s)}p_1(s)y_1(s). \]

Take collateral constraint

\[ b_1 f(s) \leq p_1(s)k(1) \]

\[ \frac{p(0)}{p_1(s)} \leq \frac{k(1)}{y(0)} + \frac{r_1(s)}{1 + r_1(s)}y_1(s) \]

which implies that

\[ \sum_s \pi(s) \frac{p(0)}{p_1(s)} \leq \frac{k(1)}{y(0)} + \sum_s \pi(s) \frac{r_1(s)}{1 + r_1(s)}y_1(s). \]

Using the no-arbitrage condition and the equilibrium value of the state price, we know that

\[ \sum_s \pi(s) \frac{p(0)}{p_1(s)} = \frac{1/\beta}{1 + r(0)} \]

Finally, we get the requirement for the collateral constraint not to bind anywhere

\[ \frac{1/\beta}{1 + r(0)} \leq \frac{k(1)}{y(0)} + \sum_s \pi(s) \frac{r_1(s)}{1 + r_1(s)}y_1(s), \quad (45) \]

That is, the expected rate of inflation must be less than the level of investment at period 0 and the expected real seigniorage cost in the second period.

The following proposition is analogous to the result in the deterministic economy, and guarantees that the collateral constraint binds at positive rates of interest, or in other words, that setting interest rates to zero is not optimal.
Proposition 3. Interest rates, \( r(0) \to 0 \) and \( r_1(s) \to 0 \), do not support prices where the collateral constraint does not bind.

Next we examine the implications of targeting a stable growth rate in prices or money supply for when the collateral constraint binds. We first make two additional assumptions:

Assumption 7. \( A_1(L) < y(0)^{1-a} \left\{ \frac{1}{\beta} - 1 \right\} \).

This ensures that the collateral constraint will bind at a positive date 0 interest rate under a policy of price stability.

Assumption 8. \( \sum_s \pi(s) A_1(s) > \beta \).

This ensures that the collateral constraint will bind at a positive date 0 interest rate under a policy of monetary stability.

Price Stability

Proposition 4. Under a policy of price stability the collateral constraint binds at a positive date 0 interest rate after a negative productivity shock.

Monetary Stability

Proposition 5. Under a policy of monetary stability the collateral constraint binds at a positive date 0 interest rate after a high productivity shock.

That the choice of policy objectives affects the type of shock under which the constraint could bind is a consequence of the interaction between the (real) productivity shock which determines the quantity of (real) capital/collateral supplied, with the nominal amount of credit available which determines the price level or money supply. For example, under price stability, a low productivity shock would require a higher quantity of state contingent bonds to be issued in order to maintain the price level, causing the collateral constraint to bind. Under monetary stability, a high productivity shock depresses the price level, and hence the nominal value of collateral, causing the collateral constraint to bind.
3.5 Binding Collateral Constraints

In this section we examine the implications for efficiency and policy when the constraint binds. That is, for some \( s^* \in S, b_1(s^*) = p_1(s^*)k(1) \). Let the states where the constraint does not bind be \( s^{**} \in S/s^8 \). In this case, we can substitute this constraint, and the constraints which the representative individual faces are

The period 0 budget constraint is then

\[
p(0)[c(0) + k(1)] \leq w(0) + q(s^*)p_1(s^*)k(1) + q(s^{**})b_1(s^{**}),
\]

The no-arbitrage condition is as before

\[
\sum_s q(s) = \frac{1}{1 + r(0)}.
\]

The transactions of the household in the second period are

\[
p_1(s^*)c_1(s^*) + p_1(s^*)k(1) \leq p(0)y(0) + \frac{p_1(s^*)y_1(s^*)}{1 + r_1(s^*)},
\]

or

\[
p_1(s^{**})c_1(s^{**}) + b_1(s^{**}) \leq p(0)y(0) + \frac{p_1(s^{**})y_1(s^{**})}{1 + r_1(s^{**})},
\]

The cash-in-advance constraints have been substituted in as has the final budget constraint, both which must bind (positive interest rate and transversality condition).

The first order conditions give us the following

\[
1 - \alpha k(1)^{a-1} \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)} = \left\{ \frac{q(s^*)p_1(s^*)}{p(0)} - \beta \pi(s^*) \right\},
\]

and

\[
q(s^{**}) = \beta \pi(s^{**}) \frac{p(0)}{p_1(s^{**})}
\]

---

8We will follow this notation for the remainder of the analysis of this two period stochastic economy.
Note that the premium generated by the constraint binding is \( q(s^*)p_1(s^*) - \beta \pi(s^*) \). With some work\(^9\) it can be shown that the premium \( \frac{q(s^*)p_1(s^*)}{p(0)} - \beta \pi(s^*) \) is positive.

### 3.5.1 Investment

**Proposition 6.** Given interest rates, \( r(0), r_1(s) \geq 0 \), investment when the collateral constraint binds is higher than if it did not.

The possibility of a binding collateral constraint in some future state introduces a premium on investment. As agents are constrained in their borrowing, any additional borrowing (above the value implied by collateral), can only increase investment. As this is better than not borrowing/investing, the equilibrium level of investment increases once the collateral constraint binds: Households over-invest compared to unconstrained borrowing case, so as to relax the collateral constraint they face. This result does not obtain in Kiyotaki and Moore, since they assume a constant marginal product of capital. Instead, households invest less than they would if they were not constrained. Our result is similar to the results of the literature on precautionary savings\(^10\), where households maintain a higher level of precautionary savings when markets are incomplete. Binding collateral constraints introduce this incompleteness as they restrict the ability to trade assets, and hence the consumption plans households can achieve intertemporally.

### 3.5.2 Policy Objectives Under a Binding Collateral Constraint

That the collateral constraint binds in equilibrium has direct implications for the ability of the monetary authority to achieve target rates of inflation or monetary growth as the following two propositions show.

**Price Stability and Inflation**

**Proposition 7.** Given interest rates \( r(0), r_1(s) \geq 0 \), and a binding collateral constraint, inflation is higher compared to the unconstrained case.

---

\(^9\)Using all the constraints in the model, one can show that as the Lagrange multiplier for the collateral constraint is positive, the liquidity premium is also positive.

Monetary Stability and Money Growth

**Proposition 8.** Given interest rates, \( r(0), r_1(s) \geq 0 \), monetary growth is higher when the collateral constraint binds is higher than if it did not.

When the collateral constraint binds, the liquidity premium generated affects the growth rate in prices and money supply, and interest rates which may have supported target rates of either are now too high. Put another way, an interest rate rule which targets a stable growth in prices or money supply may result in seemingly surprise or unannounced bouts of inflation or money growth once the collateral constraint binds.

### 3.6 Optimal Monetary Policy

In this section we characterise the optimal monetary policy. In previous section we showed that although positive interest rates are costly, setting them to zero may distort the economy via the liquidity premium associated with a binding collateral constraint. In this section, we characterise the optimal set of interest rates. Our benchmark is the level of investment, \( k(1) \), under second period interest rates that are close to zero, i.e. \( \forall s \in S, r_1(s) = r_1 \to 0 \). When the constraint does not bind, this level of investment achieves the first best allocation and it coincides with an equilibrium of economy in which households are unconstrained in their investment decisions and do not incur interest rates costs as in Lucas and Stokey. Positive interest rates distort the optimal level of investment, but are necessary to ensure (real and nominal) determinacy when the collateral constraint is expected to bind under a Friedman rule. Intuitively, positive interest rates in the second period result in lower than optimal capital investment. On the other hand, the prospect of a binding collateral constraint introduces a premium in holding more capital. Monetary policy can balance this trade-off by appropriately choosing the period 0 interest rate given positive second period interest rates. Essentially, we show that it can restore capital investment back to its efficient level even under positive interest rates. We first discuss how period 0 interest rates affect the premium associated with a binding collateral constraint under both price and monetary stability objectives. We, then, describe the optimal choice of monetary policy to balance the trade-off between under-investment due to positive interest rates, and over-investment due to a binding collateral constraint.
3.6.1 Capital Accumulation and Interest Rates

This section describes how investment responds to changes in the date 0 interest rate when the collateral constraint binds.

**Proposition 9.** Given interest rates, \( r(0), r_1(s) \geq 0 \), raising the date zero interest rate lowers the rate of capital accumulation.

3.6.2 Optimal Interest Rates

The ability of the monetary authority to control the level of capital accumulation by appropriately choosing the interest rate at date 0 and by setting positive interest rates in period 1 to determine the allocation raises the question whether it can achieve the optimal level of investment. We show that the optimal interest rate at date 0 is strictly lower than that which relaxes the collateral constraint. The solution is conditional on a given date 1 interest rate policy. Although the monetary authority distorts the efficient investment in capital by choosing positive period 1 interest rates, it can correct for this inefficiency by choosing the interest rate in the previous period such that the collateral constraint binds sufficiently enough and households choose a higher level of investment. The analysis can be easily extended in multiple periods and implies that the first best can be obtained under a interest rate policy that deviates from the Friedman rule.

**Proposition 10.** Given date one interest rates, the optimal date zero rate is strictly less than that which relaxes the collateral constraint.

3.7 Discussion of Results

When interest rates are zero, the cash in advance constraint may bind, and the nominal indeterminacy manifests itself in real indeterminacy. Typically zero nominal interest rates manifest only nominal indeterminacy. When the collateral constraint binds, the agents cannot optimally choose their asset portfolios, but are constrained by the nominal value of collateral they hold; changing nominal prices affects asset demands, which in turn affects capital accumulation decisions and real output. In effect, the binding collateral constraint obfuscates the distinction between nominal assets (state contingent bonds) and real assets (capital) thus collapsing the distinction between nominal and real effects typical in a complete markets specification. The
date 0 interest rate, played no part in the complete markets economy with a non binding collateral constrain. Since the endowment was given at date 0, there was no seigniorage cost and, since there markets were complete, date zero interest had no cost. When we remove the complete markets assumption however, this has a real effect: it increases the cost of holding money balances. To offset this the real value of debt borrowed, and hence the real value of capital/collateral needed to support this is reduced. This can be seen from propositions 7 and 8. As a result, the seigniorage cost incurred in the second period and the negative effect on capital accumulation, can be offset by choosing a rate of interest that supports a liquidity premium that offsets the seigniorage cost.

4 Infinite Horizon

The results in the previous sections extend to the infinite-horizon economies. A general argument would be to construct an equilibrium in the infinite-horizon economy as the limit of a sequence of equilibria of finite-horizon economies. Given that most macro models are set in infinite horizon, however, we provide more detailed analysis on the infinite-horizon economy. For simplicity, we consider the case of flexible prices, and show that results in Section 2 generalize to the infinite-horizon economy. We also discuss recursive equilibria and contrast our result with the one obtained by Lucas and Stokey (1987).

Suppose that shocks follow a Markov chain with transition probabilities \( \pi(s^t|s) > 0 \). The history of shocks up through date \( t \) is denoted by \( s^t = (s_0, \ldots, s_t) \), and called a date-event. The initial shock, \( s_0 \), is given, and the initial date-event is denoted by 0. The probability of date-event \( s^t|s^{t-1} \) is \( \pi(s^t|s^{t-1}) = \pi(s_t) \).

Let \( \mu \) denote the nominal equivalent martingale measure. It is a probability measure over the date-event tree with \( \mu(s^t|s^{t-1}) > 0 \), all \( s^t|s^{t-1} \). Let \( q(s^{t+1}|s^t) \) denote the price at \( s^t|s^{t-1} \) of the elementary security that pays off one unit of currency if and only if the date-event \( s^{t+1}|s^t \) is reached. Then,

\[
q(s^{t+1}|s^t) = \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t|s^{t-1})},
\]
where \( r(s^t|s^{t-1}) \) is the nominal interest rate at \( s^t|s^{t-1} \). More generally,

\[
q(s^j|s^t) = \frac{1}{1 + r(s^t|s^{t-1})} \cdots \frac{1}{1 + r(s^{j-1}|s^{j-2})} \mu(s^j|s^t),
\]

where \( q(s^j|s^t) \) is the price at \( s^t|s^{t-1} \) of the contingent claim that pays off one unit of currency if and only if \( s^j \) is reached \( (q(s^t|s^t) = 1) \).

For simplicity, consider the flexible-price economy. The representative household has preferences given by

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t f(s^t|s^{t-1}) \log c(s^t|s^{t-1}), \tag{51}
\]

where \( c(s^t|s^{t-1}) \) is consumption at \( s^t|s^{t-1} \); \( y(s^t|s^{t-1}) \) is production. At each date-event, the asset market opens first, followed by the goods market. As in Section 2, the constraints the household faces are summarized by: (i) the flow budget constraints:

\[
P(0)[c(0) + k(s^1|0)] + m(0) \leq \frac{1}{1 + r(0)} \sum_s \mu(s)b(s^1) + w(0) \tag{52}
\]

\[
P(s^t)[c(s^t|s^{t-1}) + k(s^{t+1}|s^t)] + m(s^t|s^{t-1}) + b(s^t|s^{t-1}) \leq m(s^{t-1}|s^{t-2}) + \frac{1}{1 + r(s^t|s^{t-1})} \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t)b(s^{t+1}|s^1) \tag{53}
\]

(ii) the cash constraints:

\[
m(s^t|s^{t-1}) \geq P(s^t|s^{t-1})y(s^t|s^{t-1}), \tag{55}
\]

Here, \( P(s^t|s^{t-1}) \) is the price level at \( s^t|s^{t-1} \); \( m(s^t|s^{t-1}) \) is the nominal balances carried over from \( s^t|s^{t-1} \) to the next period; \( w(s^{t+1}|s^t) \) is the nominal value of the financial asset at the beginning of \( s^{t+1}|s^t \); Remember that when the goods market follow the asset market, the cash-in-advance constraint is equivalent to (55).

The first-order conditions are given by

\[
k(s^{t+1}|s^t) = \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{y(s^{t+1}|s^t)}{1 + r(s^{t+1}|s^t)} \frac{c(s^t|s^{t-1})}{c(s^{t+1}|s^1)}, \tag{56}
\]
\[
\frac{\beta c(s^t | s^{t-1}) \pi(s^{t+1} | s^t)}{c(s^{t+1} | s^t)} = \frac{P(s^{t+1} | s^t)}{P(s^t | s^{t-1})} \frac{\mu(s^{t+1} | s^t)}{1 + r(s^t | s^{t-1})},
\]

and the transversality condition is

\[
\lim_{j \to \infty} \sum_{s^j | s^t} q(s^j | s^t) w(s^j) = 0.
\]

The flow budget constraint of the monetary-fiscal authority is

\[
\frac{r(s^t | s^{t-1})}{1 + r(s^t | s^{t-1})} M(s^t | s^{t-1}) = M(s^{t-1} | s^{t-2}) + \frac{1}{1 + r(s^t | s^{t-1})} \sum_{s^{t+1} | s^t} \mu(s^{t+1} | s^t) W(s^{t+1} | s^t),
\]

where \( W(0) = w(0) \) is given. Monetary policy sets a path of nominal interest rates, \{r(s^t | s^{t-1})\}, and a path of portfolio restrictions, \{f(s^t | s^{t-1})\}.

### 4.1 Equilibrium Analysis

At date event \( s^t \), let \( c(s^t | s^{t-1}) = \gamma A(s^t | s^{t-1}) k(s^t | s^{t-1})^\alpha \). Then (56) becomes

\[
k(s^{t+1} | s^t) = \sum_{s^{t+1} | s^t} \alpha \beta \pi(s^{t+1} | s^t) \frac{1}{1 + r(s^{t+1} | s^t)} A(s^t | s^{t-1}) k(s^t | s^{t-1})^\alpha
\]

\[
k(s^{t+1} | s^t)
\]

is measurable with respect to the information available at \( s^t \). But, \( c(s^t | s^{t-1}) + k(s^{t+1} | s^t) = A(s^t | s^{t-1}) k(s^t | s^{t-1})^\alpha \), hence,

\[
1 - \gamma = \sum_{s^{t+1} | s^t} \alpha \beta \pi(s^{t+1} | s^t) \frac{1}{1 + r(s^{t+1} | s^t)}.
\]

Households invest a fixed portion of output as (we assume) that future interest rates depend only on the the realization of the state, \( s_{t+1} \), and not on the current state, \( s^t \), i.e. \( r(s^{t+1} | s^t) = r(s_{t+1}) \).

Iterating \( c(s^t | s^{t-1}) = \gamma A(s^t | s^{t-1}) k(s^t | s^{t-1})^\alpha \) forward and substituting the expression for \( k(s^{t+1} | s^t) \), we get the following law of motion for consumption:

\[
c(s^{t+1} | s^t) = A(s^{t+1} | s^t) [1 - \gamma]^\alpha [c(s^t | s^{t-1})]^\alpha.
\]

Similarly, the law of motion for output is given by:

\[
y(s^{t+1} | s^t) = A(s^{t+1} | s^t) [1 - \gamma]^\alpha y(s^t | s^{t-1}).
\]
4.2 Equilibrium Analysis, Monetary Policy and the Collateral Constraint

In this section we show how the results of the stochastic economy regarding when the collateral constraint binds generalise over the infinite horizon.

Price Stability

Proposition 11. Given a stationary sequence of positive interest rates and a policy of price stability, the collateral constraint binds after a sequence of positive productivity shocks followed by a negative shock.

Although we do not consider an economy where the collateral constraint actually binds here, it is straightforward that a sequence of shocks such as these would cause the collateral constraint to bind. Under price stability, the sequence of shocks is reminiscent of the Minsky (1992) financial instability hypothesis.

Monetary Stability

Proposition 12. Given a stationary sequence of positive interest rates and a policy of monetary stability, the collateral constraint binds after a sequence of negative productivity shocks followed by a positive shock.

Previously we found that policies of price (respectively monetary) stability results in the collateral constraint binding after a negative (respectively positive) productivity shock. In the infinite horizon the sequence of productivity shocks under different policy objectives affects both the quantity of capital accumulated and the amount of credit available over the history. However, these can be neatly summarised by changing the date 0 endowment in the two period stochastic economy, giving qualitatively similar results.

4.3 Optimal Policy

We now turn to the issue of whether the optimal cashless allocation can be supported by a set of prices determined by monetary policy. In the following proposition, we first examine whether such a policy, if it exists, is Markovian, and secondly whether a path of interest rates exists under policy objectives of price and monetary stability, that support the optimal allocation.
Proposition 13. A sequence of positive interest rates exists which supports the first best (Pareto optimal) allocation and requires interest rates that are non-Markovian.

That an efficient path of interest rates exists is a consequence of inflation affecting only the nominal value of a predetermined level of output. In other words, the initial interest rate does not distort the level of output at date 0 gives us the extra degree of freedom to support the optimal allocation. If the economy required both capital and labour as inputs, for example, such a policy could not exist as the supply of labour in every period will be affected by the nominal interest rate, and is suboptimally low at positive rates of interest.

5 Final Remarks

We contribute to the literature on the role that financial constraints play as elements of business cycles by considering the role that monetary policy, or more precisely the path of inflation plays in determining the efficiency of market outcomes. Under complete market monetary policy can determine both when collateral constraint binds, as well as the optimality of the resulting allocation. More importantly, we show that allowing for the collateral constraint to bind may be desirable in monetary economies where a path of interest rates exists to support the first best allocation, which may be not possible in an economy without both a money and collateral constraints simultaneously. Finally, in an economy with a complete set of nominal contingent claims such as ours, the trade-off between inflation and financial frictions does not effect the allocation when monetary policy is chosen appropriately. Nevertheless, optimal monetary policy may not exist when markets are incomplete or with heterogeneous agents. In such setting there will be additional roles for fiscal policy or financial regulation, as in Goodhart et al. (2011).
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URL: [http://ideas.repec.org/p/nbr/nberwo/5684.html](http://ideas.repec.org/p/nbr/nberwo/5684.html)

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6 Appendix

6.1 Alternative Collateral Constraint

Assume that price stability prevails and the collateral constraint is instead

\[ b(1) \leq p_1(1) \frac{k(1)}{1 + r(1)}. \]  \hfill (64)

When the constraint does not bind, this can be written as

\[ \frac{b(1)}{p(0)} \beta \leq k(1) \]  \hfill (65)

using the no-arbitrage relation. This states that the value of capital must be greater than the real future value of debt.

In this setting, the first order condition for capital becomes

\[ 1 - \alpha k(1)^{a-1} \beta \frac{A_1(1)}{1 + r_1(1)} = \frac{1}{1 + r(1)} \left\{ \frac{1}{1 + r(0)} \frac{p(1)}{p(0)} - \beta \right\}. \]  \hfill (66)

The right hand side of equation 66 becomes

\[ \frac{1}{1 + r(1)} \left\{ \frac{1}{1 + r(0)} \frac{M(1) y(0)}{M(0) y(1)} - \beta \right\}. \]

The collateral constraint tells us

\[ b(1) = p(1) \frac{k(1)}{1 + r(1)} \]

\[ [M(0) - \frac{r(1)}{1 + r(1)} M(1)] = \frac{1}{1 + r(1)} \frac{M(1)}{y(1)} k(1) \]

\[ M(0) = M(1)[\frac{1}{1 + r(1)} \frac{k(1)}{y(1)} + \frac{r_1}{1 + r_1}] \]

\[ \frac{1}{1 + r(1)} \frac{k(1)}{y(1)} + \frac{r(1)}{1 + r(1)} = \frac{M(1)}{M(0)}. \]

We can substitute this in when it binds into the right hand side of equation 66 to get
\[
\frac{1}{1 + r(1)} \left\{ \frac{1}{1 + r(0)} \frac{k(1)}{1+r(1)} + \frac{r(1)}{1+r(1)} y(1) \right\} - \beta.
\]

At the optimum, \(\alpha k(1)^{\alpha-1} \beta A(1) = 1\), hence we can solve for the optimal \(\hat{r}(0)\) from

\[
\hat{r}(1) = \frac{y(0)}{r(1)+\beta} \left\{ \frac{k(1)}{1+r(1)} + \frac{r(1)}{1+r(1)} y(1) \right\}.
\]

### 6.2 Proofs

#### Proof of Proposition 1

Proof. The collateral constraint says

\[b(1) \leq p(1)k(1)\]

As \(b(1) = B(1)\) and using the monetary authority budget constraint, the left hand side of the collateral constraint becomes \(M(0) - \frac{r(1)}{1+r(1)} M(1)\). The right hand side becomes \(\frac{M(1)}{y(1)} k(1)\) once the cash-in-advance constraint is substituted in, and using market clearing. Finally, rearranging the expressions gives us

\[
\frac{M(1)}{M(0)} \geq \frac{1}{\frac{k(1)}{y(1)} + \frac{r(1)}{1+r(1)}}.
\]

Now the right hand side does not depend on \(r(0)\) and can be written as \(\frac{1}{A(1)^{\alpha-1} \left\{ \alpha \beta \frac{1}{1+r(1)} \right\}^{\alpha} + \frac{r(1)}{1+r(1)}}\). Using Assumption 2, the right hand side must be greater than \(\frac{1}{A(1)^{\alpha-1} + \frac{r(1)}{1+r(1)}}\). The left hand side does depend on \(r(0)\). The first order condition for the bond, the Fisher equation gives us: \(\frac{1}{1+r(0)} \frac{p(1)}{p(0)} = \beta\). Substituting in the cash-in-advance constraints, and using market clearing, this becomes \(\frac{M(1)}{M(0)} = \beta \frac{y(1)}{y(0)} (1 + r(0))\). As the allocation does not depend on \(r(0)\) (it is purely inflationary), the left hand side is a linear function of the date 0 interest rate, with a minimum at \(r(0)\) of \(\beta \frac{y(1)}{y(0)} = \beta \frac{A(1)k(1)^{\alpha}}{y(0)} = \)
\[
\frac{\{A(1)\beta\}^{1/(1-\alpha)}}{y(0)} \left\{ \frac{\alpha}{1+r(1)} \right\}^{\alpha/(1-\alpha)}.
\]
Using Assumption 1 this must be less than
\[
\frac{\{A(1)\beta\}^{1/(1-\alpha)}}{y(0)} \left\{ \frac{\alpha}{1+r(1)} \right\}^{\alpha/(1-\alpha)} = 1 + \frac{r(1)}{\alpha}.
\]
Finally Assumption 3 guarantees that this is less than \(\frac{1}{A(1)\alpha - 1 + \frac{r(1)}{1+r(1)}}\), hence there will be a positive interest rate at which the constraint will first be relaxed.

**Proof of Proposition 2**

**Proof.** Taking the total utility in the constrained economy

\[
U = y(0) - k(1) + \beta Ak(1)^{\alpha}
\]

\[
\frac{\partial U}{r(0)} = - \frac{\partial k(1)}{r(0)} \{1 - \alpha \beta Ak(1)^{\alpha-1}\}.
\]

The optimal date zero interest rate, \(\hat{r}(0)\), occurs when \(\alpha \beta Ak(1)^{\alpha-1} = 1\). Substituting this, and using equation 21 we can find the date 0 interest rate which achieves the first-best (Pareto optimal) allocation:

\[
\hat{r}(0) = \frac{\frac{y(0)}{k(1)+A(1)k(1)^{\alpha} \frac{r(1)}{1+r(1)}}}{\beta + \frac{r(1)}{1+r(1)}} - 1.
\]

We can compare this with the interest rate which supports the allocation when the constraint binds at the margin,

\[
b(1) = p(1)k(1)
\]

\[
p(0)y(0) - p(1) = y(1) \frac{r(1)}{1+r(1)} \leq p(1)k(1)
\]

\[
\frac{y(0)}{k(1) + y(1) \frac{r(1)}{1+r(1)}} \leq \frac{p(1)}{p(0)}
\]

\[
\frac{y(0)}{k(1) + y(1) \frac{r(1)}{1+r(1)}} \leq \beta(1 + r(0)).
\]

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\[
\bar{\tau}(0) = \frac{y(0)}{k(1 + A(1)k(1)^\alpha \frac{\tau(1)}{\tau(0)})} - 1
\]
\[> \hat{r}(0).\]

The last step follows from the result that in the constrained economy the level of capital accumulation is decreasing on date zero interest rates. \(\square\)

**Proof of Proposition 3**

*Proof.* For the first part, take all the interest rates in equation 45 to zero. This gives

\[
1/\beta \leq \frac{k(1)}{y(0)} \tag{71}
\]

Note that

\[
\frac{k(1)}{y(0)} = \frac{\{\alpha \beta \sum_s \pi(s) A_1(s)\}^{1/(1-\alpha)}}{y(0)}
\]

Now let \(\beta \to 0\). The left hand side of 71 approaches infinity while the right hand side approaches zero hence they must cross for some level of \(\beta\). In general it is easy to see that there are enough degrees of freedom for the parameters in the problem such that the collateral constraint will not be satisfied. \(\square\)

**Proof of Proposition 4**

*Proof.* Let the monetary authority choose portfolio weights \(f(s)\) such that \(\forall s \in S, p_1(s) = p_1\). Take first order condition for asset trade for riskless portfolio: \(\frac{p_1}{p(0)} = \beta(1 + r(0))\). Consider for some state \(s' \in S\), the collateral constraint:
\[ b_1(s') \leq p_1(s')k(1) \]
\[ p(0)y(0) - p_1(s')y_1(s') \frac{r_1(s')}{1 + r_1(s')} \leq p_1(s')k(1) \]
\[ \frac{y(0)}{k(1) + y_1(s') \frac{r_1(s')}{1 + r_1(s')}} \leq \frac{p_1(s')}{p(0)} \]
\[ \frac{y(0)}{k(1) + y_1(s') \frac{r_1(s')}{1 + r_1(s')}} \leq \beta(1 + r(0)). \] (72)

The left hand side is independent of the date 0 interest rate. Using assumption 5 it must be greater than \( \frac{1}{1 + A_1(s'')y(0)^{\alpha - 1} \frac{r_1(s')}{1 + r_1(s')}} \). Assumption 4 gives us that this is highest when the productivity shock is lowest. The right hand side is linearly increasing on \( r(0) \) and has a minimum at \( \beta \). Hence if the constraint is to bind then
\[ \frac{1}{1 + A_1(s')y(0)^{\alpha - 1} \frac{r_1(s')}{1 + r_1(s')}} > \beta \]
\[ A_1(s') \frac{r_1(s')}{1 + r_1(s')} < y(0)^{1 - \alpha} \left\{ \frac{1}{\beta} - 1 \right\} \]

The left hand side has a minimum at \( A_1(L) \), and using assumption 7 the above inequality is satisfied. Hence under a policy of price stability, the collateral constraint binds after a negative productivity shock. Furthermore, the collateral constraint binds at a positive date 0 interest rate.

**Proof of Proposition 5**

*Proof.* In this case, \( \forall s \in S, M_1(s) = M_1(s) \). The no-arbitrage condition gives us \( \frac{M(0)}{M_1} \sum_s \beta^{y_1(s)} \frac{y(0)}{y(0)} = \frac{1}{1 + r(0)} \) or \( (1 + r(0)) \sum_s \beta^{y_1(s)} y(0) = \frac{M_1}{M(0)} \). Consider the collateral constraint:
\begin{align*}
b_1(s') &\leq p_1(s') k(1) \\
M(0) - \frac{r_1(s')}{1 + r_1(s')} M_1(s') &\leq M_1(s') \frac{k(1)}{y_1(s')} \\
\frac{1}{y_1(s')} + \frac{r_1(s')}{1 + r_1(s')} &\leq \frac{M_1(s)}{M(0)} \\
&\leq (1 + r(0)) \sum_s \frac{\pi(s)y_1(s)}{y(0)} ,
\end{align*}

\[
\sum_s \frac{\pi(s)y_1(s)}{y(0)} \left\{ \frac{k(1)}{y_1(s')} + \frac{r_1(s')}{1 + r_1(s')} \right\} \leq (1 + r(0)) \beta.
\]

The right hand side is minimised at \( r(0) = 0 \). Without loss of generality we can set \( r_1(s') \to 0 \) as this minimises the cost of seigniorage on the optimal investment decision. The inequality if the constraint is to bind then is

\[
\sum_s \frac{\pi(s)y_1(s)}{y(0)} \frac{k(1)}{A_1(s')} \sum_s \pi(s)A_1(s) < \frac{\sum_s \pi(s)A_1(s)}{A_1(s')}
\]

where the last step comes from assumption 5. Finally note that from assumption 4, this is smallest following a high productivity shock. The condition which guarantees that the constraint binds here is assumption 8:

\[
\sum_s \frac{\pi(s)A_1(s)}{A_1(H)} > \beta.
\]

Hence if the collateral constraint is to bind, it will be when there is a positive productivity shock. Additionally, there exists a positive date 0 interest rate at which the constraint first binds.

\section*{Proof of Proposition 6}

\textit{Proof.} From 49

\[1 - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)} > 0.\]

If the constraint did not bind, it would be zero. As \( \alpha < 1 \), it follows that investment is higher when the collateral constraint binds.
Proof of Proposition 7

Proof. Let $s \in S, p_1(s) = p_1$ and combine 49 and 50

\[
\sum_s q(s) = \frac{p(0)}{p_1(s^*)} \left[ 1 + \beta \pi(s^*) - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right] + \sum_{s^* \in S} \beta \pi(s^*) \frac{p(0)}{p_1(s^*)}
\]

\[
\frac{1}{1 + r(0)} = \frac{p(0)}{p_1} \left[ \beta + 1 - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right]
\]

\[
\frac{p_1}{p(0)} = (1 + r(0)) \beta \left[ 1 + \frac{1}{\beta} \left\{ 1 - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right\} \right].
\]

(73)

As the rate of inflation when the constraint does not bind is $(1 + r(0)) \beta$, it follows that inflation is higher when the constraint binds than when it does not.

Proof of Proposition 8

Proof. Let $s \in S, M_1(s) = M_1$ and combine 49 and 50

\[
\sum_s q(s) = \frac{p(0)}{p_1(s^*)} \left[ 1 + \beta \pi(s^*) - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right] + \sum_{s^* \in S} \beta \pi(s^*) \frac{p(0)}{p_1(s^*)}
\]

\[
\frac{1}{1 + r(0)} = \frac{M(0)}{M_1} \left\{ \frac{y_1(s^*)}{y(0)} \left[ 1 + \beta \pi(s^*) - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right] + \beta \pi(s^*) \frac{y_1(s^*)}{y(0)} \right\}
\]

\[
\frac{M_1}{M(0)} = (1 + r(0)) \beta \left[ \sum_s \frac{y_1(s)}{y(0)} \pi(s) + \frac{y_1(s^*)}{y(0)} \beta \left\{ 1 - \alpha k(1)^{\alpha - 1} \sum_s \frac{\pi(s) A_1(s)}{1 + r_1(s)} \right\} \right].
\]

(74)

As the rate of money growth when the constraint does not bind is

\[
(1 + r(0)) \beta \sum_s \frac{y_1(s)}{y(0)} \pi(s),
\]

(75)

it follows that money growth is higher when the constraint binds than when it does not.
Proof of Proposition 9

Proof. Price Stability Consider equation 49

\[ 1 - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)} = \left\{ \frac{q(s^*)p_1(s^*)}{p(0)} - \beta \pi(s^*) \right\} \]

Using \( q(s^*) = \frac{1}{1 + r(0)} - \sum_{s^* \in S/s^*} \beta \pi(s^*) \frac{p(0)}{p_1(s^*)} \), the right hand side can be written as:

\[ \frac{1}{1 + r(0)} - \sum_{s^* \in S/s^*} \beta \pi(s^*) \frac{p_1(s^*)}{p_1(s^*)} - \beta \pi(s^*). \]

Under price stability, the monetary authority will choose its portfolio such that \( \forall s \in S, p_1(s) = p_1 \), thus the equation becomes:

\[ \frac{1}{1 + r(0)} - \beta. \]

Moreover, \( p_1(s^*) = \frac{M_1(s^*)}{y_1(s^*)}, p(0) = \frac{M(0)}{y(0)} \) and \( y_1(s^*) = A_1(s^*)k(1)^a \), thus the right hand side is:

\[ \frac{1}{1 + r(0)} - \beta. \]

Consider the binding collateral constraint for state \( s^* \):

\[ b_1(s^*) = p_1(s^*)k(1) \]

\[ M(0) - \frac{r_1}{1 + r_1} M_1(s^*) = \frac{M_1(s^*)}{y_1(s^*)} k(1) \]

\[ M(0) = M_1(s^*) \left( \frac{k(1)}{y_1(s^*)} + \frac{r_1}{1 + r_1} \right) \]

\[ \frac{M_1(s^*)}{M(0)} = \frac{k(1)}{y_1(s^*)} + \frac{r_1}{1 + r_1} \]

Substituting the above equation, 49 becomes:

\[ 1 - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)} = \frac{1}{1 + r(0)} \frac{1}{k(1)} \frac{y(0)}{y_1(s^*)} + \frac{r_1}{1 + r_1} - \beta \]

\[ 1 - \alpha k(1)^{\alpha - 1} \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)} = \frac{1}{1 + r(0)} y(0) \frac{k(1)}{k(1) + A_1(s^*)k(1)^a \frac{r_1}{1 + r_1}} - \beta \]

(76)

Total differentiation with respect to \( r(0) \) yields:

\[ \frac{\partial k(1)}{\partial r(0)} \left[ \alpha (1 - \alpha) k(1)^{\alpha - 2} \beta \sum_s \frac{\pi(s)A_1(s)}{1 + r_1(s)} \right] \]

\[ = -\frac{1}{(1 + r(0))^2} k(1) + A_1(s^*)k(1)^{1-a} - \frac{1}{1 + r(0)} y(0) \frac{\partial k(1)}{\partial r(0)} \left( k(1) + A_1(s^*)k(1)^{a} \frac{r_1}{1 + r_1} \right)^2 \]

47
Collecting the terms for \( \frac{\partial k(1)}{\partial r(0)} \) we find that \( \frac{\partial k(1)}{\partial r(0)} < 0 \). That is, raising the date zero interest rate lowers the rate of capital accumulation. The monetary authority can choose a high enough period 0 interest rate to fully relax the collateral constraint. Conversely, it can choose lower levels of \( r(0) \) such that the constraint is more binding and capital investment higher.

**Monetary Stability** Under a policy of Monetary Stability the monetary authority and equal interest states in the intermediate period, \( \forall s \in S \), money supplies \( M_1(s) = M_1 \). Thus, the right hand side of equation 49 becomes:

\[
\frac{1}{1 + r(0)} \frac{p_1(s^*)}{p(0)} - \sum_{s^{**} \in S/s^*} \beta \pi(s^{**}) \frac{p_1(s^*)}{p_1(s^{**})} - \beta \pi(s^*) \\
= \frac{1}{1 + r(0)} \frac{M_1(s^*)}{M(0)} \frac{y(0)}{y_1(s^*)} - \sum_{s^{**} \in S/s^*} \beta \pi(s^{**}) \frac{M_1(s^*)}{M_1(s^{**})} \frac{A_1(s^{**})}{A_1(s^*)} - \beta \pi(s^*) \\
= \frac{1}{1 + r(0)} \frac{M_1(s^*)}{M(0)} \frac{y(0)}{y_1(s^*)} - \sum_{s^{**} \in S/s^*} \beta \pi(s^{**}) \frac{A_1(s^{**})}{A_1(s^*)} - \beta \pi(s^*).
\]

Using the collateral constraint we get

\[
\frac{1}{1 + r(0)} \frac{y(0)}{k(1) + \frac{r_1}{1+r_1} y_1(s^*)} - \sum_{s^{**} \in S/s^*} \beta \pi(s^{**}) \frac{A_1(s^{**})}{A_1(s^*)} - \beta \pi(s^*). \tag{77}
\]

In addition, the second and third terms are constants, thus they will drop in the total differentiation with respect to \( r(0) \). As a result, the analysis is as before and \( \frac{\partial k(1)}{\partial r(0)} < 0 \). The monetary authority can relax (tighten) the collateral constraint by choosing high (low) period 0 interest rates and, thus, it can control the level of investment.

**Proof of Proposition 10**

Proof. Taking the total utility in the constrained economy

\[
U = y(0) - k(1) + \beta k(1)^\alpha \sum_s \pi(s) A_1(s) \tag{78}
\]

\[
\frac{\partial U}{r(0)} = - \frac{\partial k(1)}{r(0)} \{1 - \alpha \beta k(1)^{\alpha-1} \sum_s \pi(s) A_1(s)\}. \tag{79}
\]

48
Under Price Stability

For simplicity and without loss of generality, assume that \( \forall s \in S, r_1(s) = r_1 \). The optimal date zero interest rate, \( \hat{r}(0) \), occurs when \( \alpha \beta k(1)^{\alpha-1} \sum_s \pi(s)A_1(s) = 1 \). Using equation 76:

\[
\hat{r}(0) = \frac{y(0)}{k(1)+A_1(s^*)k(1)^{\alpha} \frac{1+r_1}{1+r_1}} - 1. \tag{80}
\]

When the constraint just binds, from equation 72,

\[
\tau(0) = \frac{y(0)}{k(1)+r_1A_1(s^*)k(1)^{\alpha} \frac{1+r_1}{1+r_1}} - 1 > \hat{r}(0).
\]

The last step follows from the result that in the constrained economy the level of capital accumulation is decreasing on date zero interest rates.

Under Monetary Stability Using 77, equation 49 becomes

\[
1 - \alpha k(1)^{\alpha-1} \beta \sum_s \pi(s)A_1(s) = \frac{1}{1 + r(0)} \frac{y(0)}{k(1)+r_1A_1(s^*)} - \sum_{s^* \in S/s^*} \beta \pi(s^{**}) \frac{A_1(s^{**})}{A_1(s^*)} - \beta \pi(s^*).
\]

As before, at the optimal interest rate \( \alpha \beta k(1)^{\alpha-1} \sum_s \pi(s)A_1(s) = 1 \) so the above becomes

\[
\frac{1}{1 + \hat{r}(0)} \frac{y(0)}{k(1)+r_1A_1(s^*)} = \frac{r_1(s)}{1 + r_1(s)} + \frac{\beta \sum_s \pi(s)y_1(s)}{A_1(s^*)k(1)^{\alpha}}
\]

\[
\hat{r}(0) = \frac{y(0)}{r_1 \frac{1+r_1}{1+r_1} + \frac{\beta \sum_s \pi(s)y_1(s)}{y_1(s^*)}} - 1. \tag{81}
\]

The interest rate at which the constraint just binds occurs when the rate of money growth is given by equation 75. Using the collateral constraint, it turns out that the interest rate which supports this is

\[
\tau(0) = \frac{y(0)}{r_1 \frac{1+r_1}{1+r_1} + \frac{\beta \sum_s \pi(s)y_1(s)}{y_1(s^*)}} - 1. \tag{82}
\]
The result is thus the same: given date 1 interest rates, the optimal interest rate allows for the collateral constraint to bind.

Proof of Proposition 12

Proof. The banks' portfolio problem yields the following relation between bond prices and the interest rate at state $s^t|s^{t-1}$

$$\sum_{s^t|s^{t-1}} q(s^{t+1}|s^t) = \frac{1}{1 + r(s^t|s^{t-1})}$$

(83)

But, individual optimization with respect to bond holding $b(s^{t+1}|s^t)$ yields

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t) \frac{c(s^{t+1}|s^t)}{c(s^t|s^{t-1})} \frac{P(s^t|s^{t-1})}{P(s^{t+1}|s^t)}$$

(84)

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t) \frac{c(s^{t+1}|s^t)}{c(s^t|s^{t-1})} \frac{y(s^{t+1}|s^t) M(s^t|s^{t-1})}{y(s^t|s^{t-1}) M(s^{t+1}|s^t)}$$

(85)

$$q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t) \frac{M(s^t|s^{t-1})}{M(s^{t+1}|s^t)}$$

(86)

Summing over all bond prices we get that the money growth is given by

$$\frac{M(s^{t+1}|s^t)}{M(s^t|s^{t-1})} = \beta(1 + r(s^t|s^{t-1}))$$

(87)

For the analysis that follows it is convenient to express $k(s^{t+1}|s^t)$ in terms of period 0 production and the sequence of productivity shocks.

$$k(s^1|0) = (1 - \gamma)y(0)$$

$$k(s^2|s^1) = (1 - \gamma)A(s^1|0)((1 - \gamma)y(0))^\alpha$$

$$k(s^3|s^2) = (1 - \gamma)A(s^2|s^1)((1 - \gamma)A(s^1|0)((1 - \gamma)y(0))^\alpha)^\alpha$$

$$= [1 - \gamma]^{1+\alpha+\alpha^2} [y(0)]^{\alpha^2}$$

$$k(s^{t+1}|s^t) = [1 - \gamma]^{1+\alpha+\alpha^t} [y(0)]^{\alpha^t} \prod A(s^i|s^{i-1})$$
The nominal value of capital, hence collateral is

\[ P(s^{t+1}|s^t)k(s^{t+1}|s^t) = M(s^{t+1}|s^t) \frac{k(s^{t+1}|s^t)}{y(s^{t+1}|s^t)} \]

\[ = M(s^{t+1}|s^t) \left(1 - \gamma \right)^{1 + \alpha + \ldots + \alpha^t} \left[ y(0) \right]^{\alpha^t} A(s^t|s^{t-1})^{1-a} \]

\[ = M_0 \prod \beta \left( 1 + r(s^t|s^{t-1}) \right) \left(1 - \gamma \right)^{1 + \alpha + \ldots + \alpha^t} \left[ y(0) \right]^{\alpha^t} A(s^t|s^{t-1})^{1-a} \]

The money supply at period 0 can be derived from the present-value budget constraint of the monetary authority using the Markovian property of interest rates:

\[ \frac{r_0}{1 + r_0} M_0 + \sum_{s^t|0} q(s^t|0) \frac{r(s^t|0)}{1 + r(s^t|0)} M(s^t|0) + \sum_{s^t|0} q(s^t|0) \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) \frac{r(s^{t+1}|s^t)}{1 + r(s^{t+1}|s^t)} M(s^{t+1}|s^t) + \ldots = W_0 \]

\[ \Rightarrow \frac{r_0}{1 + r_0} M_0 + \sum_{s^t|0} \beta f(s^t|s^0) \frac{M_0}{M(s^t|0)} \frac{r(s^t|0)}{1 + r(s^t|0)} M(s^t|0) \]

\[ + \sum_{s^t|0} \beta f(s^t|s^0) \frac{M_0}{M(s^t|0)} \sum_{s^{t+1}|s^t} \beta f(s^{t+1}|s^0) \frac{M(s^{t+1}|0)}{M(s^{t+1}|s^1)} \frac{r(s^{t+1}|s^1)}{1 + r(s^{t+1}|s^1)} M(s^{t+1}|s^1) \ldots = W_0 \]

\[ \Rightarrow M_0 \left[ \frac{r_0}{1 + r_0} + (\beta + \beta^2 + \ldots) \sum_{s^t|0} f(s^t|s^0) \frac{r(s^t|0)}{1 + r(s^t|0)} \right] = W_0 \]

\[ \Rightarrow M_0 = \frac{W_0}{\frac{r_0}{1 + r_0} + \beta + \beta^2 + \ldots \sum_{s^t|0} f(s^t|s^0) \frac{r(s^t|0)}{1 + r(s^t|0)} } \]

The asset demand is increasing at a constant rate, independently of the productivity shocks. In particular, iterating the budget constraints of the monetary authority we find that:

\[ B(s^{t+1}|s^t) = M_0 \beta \left( 1 + \Psi \prod r(s^{t-1}|s^{t-2}) \right) \]

Where \( \Psi \) depends only on future (predetermined) interest rates and the discount factor \( \beta \).

Thus, the asset demand on the left hand side of the constraint is independent of the productivity shocks, while the value of collateral on the right hand side is not. In particular, the value of collateral at any point time is minimized is a series of bad productivity shocks, \( A(s^t|0), \ldots, A(s^{t+1}|s^{t-1}) \) is followed by a a good productivity shock, \( A(s^{t+1}|s^t) \), in the state that the constraint binds. Equivalently to avoid the collateral constraint, given \( \alpha \) and \( \beta \), interest rates should be high enough to create inflation. \( \square \)
Proof of Proposition 11

Proof. Under price stability, the value of the collateral is the same irrespective of the productivity realization in the future. Thus, the collateral will first bind in the state against which households have borrowed the most. It is easy to show that the value of the collateral is decreasing on the number of bad productivity shocks in the past. Using the Fisher equation and the law of motion for capital investment one can get the following expression for the evolution of inflation:

\[
\frac{P(s_{t+1}|s_t)}{P(s_t|s_{t-1})} = \beta(1+r(s_t|s_{t-1}))A(s_t|s_{t-1}) \left[(1-\gamma)A(s_t|s_{t-1})k(s_t|s_{t-1})\right]^{a(1-a)} \sum_{s_{t+1}|s_t} \frac{1}{A(s_{t+1}|s_t)}
\]

Denote the RHS of the above equation by \(\Gamma(s_t|s_{t-1})\). The realization of the future productivity shock does not affect the price nor the quantity of collateral: only the expected distribution of (Markovian) productivity shocks matters.

However, the the value of collateral, \(P(s_{t+1}|s_t)k(s_{t+1}|s_t)\), attains its minimum value at \(s_t\) if a path of bad shocks have realized in the bath. Observe from \(k(s_{t+1}|s_t) = [1-\gamma]^{1+\alpha_1+\ldots+\alpha_T}y(0)^{\alpha_T} \prod A(s_t|s_{t-1})\) that capital investment is minimized after a sequence of bad shocks. Also, iterating 88 backwards we get

\[
P(s_{t+1}|s_t) = A_0 \sum_{s_{t+1}|s_t} \frac{1}{A(s_{t+1}|s_t)} \prod \beta(1+r(s_t|s_{t-1})) \left[(1-\gamma)A(s_t|s_{t-1})k(s_t|s_{t-1})\right]^{a(1-a)}
\]

Hence, the price is minimized at a path of bad realizations as well.

If the asset demand is decreasing for a path of bad realization, then the constraint may not bind. The asset demand \(B(s_{t+1}|s_t)\) depends both on the future expected path of demands and the realization of past money supplies. In order to determine how the asset demand in the collateral constraint depends on past and current productivity realizations we iterate forward the
budget constraint of the monetary authority:

\[
M(s^{t+1}|s^t) + B(s^{t+1}|s^t) = M(s^t|s^{t-1}) + \sum_{s^{t+2}|s^{t+1}} q(s^{t+2}|s^{t+1})B(s^{t+2}|s^{t+1})
\]

\[
B(s^{t+1}|s^t) = M(s^t|s^{t-1}) - M(s^{t+1}|s^t)
\]

\[
+ \sum_{s^{t+2}|s^{t+1}} q(s^{t+2}|s^{t+1})[M(s^{t+1}|s^t) - M(s^{t+2}|s^{t+1})]
\]

\[
+ \sum_{s^{t+2}|s^{t+1}} q(s^{t+2}|s^{t+1}) \sum_{s^{t+3}|s^{t+2}} q(s^{t+3}|s^{t+2})[M(s^{t+2}|s^{t+1}) - M(s^{t+3}|s^{t+2}) + ...]
\]

\[
= M(s^t|s^{t-1}) - M(s^{t+1}|s^t) + \frac{M(s^{t+1}|s^t)}{1 + r(s^{t+1}|s^t)} - \beta M(s^{t+1}|s^t)
\]

\[
+ \sum_{s^{t+2}|s^{t+1}} q(s^{t+2}|s^{t+1})\left[\frac{M(s^{t+2}|s^{t+1})}{1 + r(s^{t+2}|s^{t+1})} - \beta M(s^{t+2}|s^{t+1}) + ...ight]
\]

\[
= M(s^t|s^{t-1}) - M(s^{t+1}|s^t) \left\{1 + \frac{1}{1 + r(s^{t+1}|s^t)} - \beta + \beta \sum_{s^{t+2}|s^{t+1}} \frac{\pi(s^{t+2}|s^{t+1})}{1 + r(s^{t+2}|s^{t+1})} - \beta^2 + ...ight\}
\]

\[
= M(s^t|s^{t-1}) - M(s^{t+1}|s^t)\Delta(s^{t+1}|s^t)
\]

Rearranging the collateral constraint gives

\[
B(s^{t+1}|s^t) \leq P(s^{t+1}|s^t)k(s^{t+1}|s^t)
\]

\[
M(s^t|s^{t-1}) - M(s^{t+1}|s^t)\Delta(s^{t+1}|s^t) \leq M(s^{t+1}|s^t)\frac{k(s^{t+1}|s^t)}{y(s^{t+1}|s^t)}
\]

\[
\frac{1}{\Delta(s^{t+1}|s^t) + \frac{k(s^{t+1}|s^t)}{y(s^{t+1}|s^t)}} \leq \frac{M(s^{t+1}|s^t)}{M(s^t|s^{t-1})}
\]

\[
\frac{1}{\Delta(s^{t+1}|s^t) + \frac{k(s^{t+1}|s^t)}{y(s^{t+1}|s^t)}} \leq \frac{P(s^{t+1}|s^t) y(s^{t+1}|s^t)}{P(s^t|s^{t-1}) y(s^t|s^{t-1})}
\]

\[
\frac{y(s^t|s^{t-1})}{y(s^{t+1}|s^t)} \Delta(s^{t+1}|s^t) + \frac{k(s^{t+1}|s^t)}{y(s^{t+1}|s^t)} \leq \Gamma(s^t|s^{t-1})
\]

Recall that \( k(s^{t+1}|s^t) = (1 - \gamma)y(s^t|s^{t-1}) \). The LHS, when rearranged, becomes:

53
\[
\frac{k(s^{t+1}|s^t)}{(1-\gamma)y(s^{t+1}|s^t)} = \frac{1}{1-\gamma} \Delta(s^{t+1}|s^t) \frac{y(s^{t+1}|s^t)}{k(s^{t+1}|s^t)} + 1
\]

\[
= \frac{1}{1-\gamma} \Delta(s^{t+1}|s^t) \frac{A(s^{t+1}|s^t)}{k(s^{t+1}|s^t)^{1-\alpha}} + 1
\]

Note that this is maximised, and hence, under a policy of price stability, the collateral constraint, if it binds, will be after a sequence of positive shocks followed by a negative one. \(\square\)

**Proof of Proposition 13**

*Proof.* Positive interest rates distort the investment decision and result in inefficiently low capital investment. The intertemporal optimality condition for capital accumulation is an unconstrained economy is:

\[
k(s^{t+1}|s^t) = \left[ \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) A(s^t|s^{t-1}) k(s^t|s^{t-1}) \right]^\alpha
\]  

(89)

Initially, we examine whether a Markovian interest rate policy can support the cashless allocation. A Markovian interest rates policy determines interest rates only on the information about the realized productivity shock and not on past realizations.

Without loss of generality, assume that the collateral constraint binds only in one state indexed by \(s^*\). The first order condition with respect to \(k(s^{t+1}|s^t)\) is

\[
- \frac{1}{c(s^t|s^{t-1})} + \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{A(s^{t+1}|s^t) k(s^{t+1}|s^t)^{\alpha-1}}{1 + r(s^{t+1}|s^t)} + \frac{1}{c(s^{t+1}|s^t)} \lambda(s^*|s^t)p(s^*|s^t) = 0
\]  

\[-1 + \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{A(s^{t+1}|s^t) k(s^{t+1}|s^t)^{\alpha-1}}{1 + r(s^{t+1}|s^t)} \frac{c(s^t|s^{t-1})}{c(s^{t+1}|s^t)} + \lambda(s^*|s^t)c(s^t|s^{t-1})p(s^*|s^t) = 0
\]  

(90)

The first order equation for the state-contingent bond which is constrained
Substituting this into the first order equation for capital gives

$$ q(s^t|s^{t-1}) = \frac{\beta \pi(s^{t+1}|s^t) P(s^t|s^{t-1}) c(s^t|s^{t-1})}{P(s^{t+1}|s^t)c(s^{t+1}|s^t)} + \frac{P(s^t|s^{t-1})c(s^t|s^{t-1})\lambda(s^t|s^t)}{P(s^{t+1}|s^t)c(s^{t+1}|s^t)} - \sum_{s^{t+1}|s^t} \frac{\beta \pi(s^{t+1}|s^t) P(s^t|s^{t-1}) c(s^t|s^{t-1})}{P(s^{t+1}|s^t)c(s^{t+1}|s^t)} \cdot \frac{1}{1 + r(s^t|s^{t-1})} \cdot \frac{\sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) P(s^t|s^{t-1}) c(s^t|s^{t-1})}{P(s^{t+1}|s^t)c(s^{t+1}|s^t)}. $$

Substituting this into the first order equation for capital gives

$$ 1 - \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{A(s^{t+1}|s^t) k(s^{t+1}|s^t)^{\alpha-1} c(s^t|s^{t-1})}{1 + r(s^{t+1}|s^t)} \cdot \frac{1}{P(s^t|s^{t-1})} \cdot \frac{1 + r(s^t|s^{t-1})}{1 + r(s^{t+1}|s^t)} \cdot \frac{\sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) P(s^t|s^{t-1}) c(s^t|s^{t-1})}{P(s^{t+1}|s^t)c(s^{t+1}|s^t)} \cdot \frac{1}{1 - \gamma}. $$

Now, the optimal, cashless allocation follows the following rules: $c(s^t|s^{t-1}) = \gamma y(s^t|s^{t-1})$ and $k(s^{t+1}|s^t) = (1 - \gamma) y(s^t|s^{t-1})$. Substituting this in gives

$$ 1 - \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{1 - \gamma}{1 + r(s^{t+1}|s^t)} \cdot \frac{1}{P(s^t|s^{t-1})} \cdot \frac{1 + r(s^t|s^{t-1})}{1 + r(s^{t+1}|s^t)} \cdot \frac{\sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) P(s^t|s^{t-1}) y(s^t|s^{t-1})}{P(s^{t+1}|s^t) y(s^{t+1}|s^t)} = (91). $$

**Under Price Stability** Under such a policy rule, equation 91 becomes

$$ 1 - \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{1 - \gamma}{1 + r(s^{t+1}|s^t)} = \frac{P(s^t|s^{t-1})}{1 + r(s^t|s^{t-1})} \cdot \frac{\sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) y(s^t|s^{t-1})}{y(s^{t+1}|s^t)}. $$

The collateral constraint under price stability is:

$$ \frac{1}{\Delta(s^t|s^t)} \frac{A(s^t|s^t)}{k(s^{t+1}|s^t)^{1-\alpha}} \cdot \frac{1}{P(s^t|s^{t-1})} = \frac{P(s^t|s^{t-1})}{P(s^t|s^{t-1})}. $$

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Note that \( \hat{\Delta}(s^*|s^t) \) depends on the expected allocation in the future and will incorporate the expected premium if the constraint is expected to bind in subsequent periods.

\[
1 - \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{1}{1 + r(s^{t+1}|s^t)} = \frac{1}{1 - \gamma} \frac{1}{1 + r(s^t|s^{t-1})} \Delta(s^*|s^t) + 1 - \sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) y(s^t|s^{t-1}) y(s^{t+1}|s^t)
\]

Finally, using the optimal allocation, we can find a rule for \( r(s^t) \) that depends on the expected path of interest rates and the optimal allocation. As we have satisfied all the equilibrium conditions, such a policy rule will support the optimal allocation, provided all interest rates are strictly positive.

**Under Monetary Stability** Targeting Monetary Stability changes equation 91 to be

\[
1 - \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{1}{1 + r(s^{t+1}|s^t)} = \frac{1}{1 - \gamma} \frac{1}{1 + r(s^t|s^{t-1})} M(s^*|s^t) M(s^t|s^{t-1})
\]

\[
\sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) M(s^t|s^{t-1})
\]

\[
= \frac{P(s^*|s^t)}{1 + r(s^t|s^{t-1})} - \frac{y(s^t|s^{t-1})}{y(s^*|s^t)} \frac{M(s^t|s^{t-1})}{y(s^t|s^{t-1})}.
\]

The collateral constraint under Monetary Stability is:

\[
\frac{1}{1 - \gamma} \frac{A(s^*|s^t)}{k(s^{t+1}|s^t)^{1 - \alpha}} + 1 = \frac{P(s^*|s^t)}{P(s^t|s^{t-1})},
\]

where \( \hat{\Delta}(s^*|s^t) \) depends on the expected allocation in the future and will incorporate the expected premium if the constraint is expected to bind in subsequent periods. Substituting this gives

\[
1 - \sum_{s^{t+1}|s^t} \alpha \beta \pi(s^{t+1}|s^t) \frac{1}{1 + r(s^{t+1}|s^t)} \frac{1}{1 - \gamma} \frac{A(s^*|s^t)}{k(s^{t+1}|s^t)^{1 - \alpha}} + 1 - \sum_{s^{t+1}|s^t} \beta \pi(s^{t+1}|s^t) y(s^t|s^{t-1}) y(s^{t+1}|s^t). \]
Finally, using the optimal allocation, we can find a rule for $r(s^t|s^{t-1})$ that depends on the expected path of interest rates and the optimal allocation. As we have satisfied all the equilibrium conditions, such a policy rule will support the optimal allocation, provided all interest rates are strictly positive. □