Knowledge-Belief Space Approach to Robust Implementation

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Abstract

In this paper, we give a characterization of robust implementation, which was first studied by Bergemann-Morris [6]. Our method is different and more general than Bergemann-Morris. We consider Bayesian implementation on the universal type space a la Mertens-Zamir. However, Mertens-Zamir’s space is not applicable here due to the existence of redundant types and the failure of Equilibrium Extension Property by Friedenberg-Meier [12]. To deal with redundancy, we adopt an extended belief hierarchy space introducing a payoff irrelevant parameter space constructed by Yokotani [23] which allows Harsanyi type spaces with redundant types to be embedded. In addition, by introducing knowledge partition to Harsanyi type spaces, we construct a "universal" type space where Equilibrium Extension Property holds. As a result, we obtain a characterization result about robust implementation by applying the methods by Jackson [13] and Palfrey-Srivastava [21] on this space. Due to the simplicity of the structure, we can easily extend this result to social choice correspondences and noisy signal models which were not covered by Bergemann-Morris.

1 Introduction

In the history of economics and game theory, it has been a central issue how to achieve a targeted resource allocation in case the social planner does not know about agents’ characteristics such as preferences or endowment. In the fields of implementation theory and more generally mechanism
design theory, they have been studying what kind of mechanism achieves social choice correspondences, that is, targeted allocations, and when there exists such mechanisms. Among them, the most important finding was the characterization of Nash implementation by Maskin [18]. He gave a necessary condition and a sufficient condition concerning when there exists a mechanism to implement a social choice correspondence as its Nash equilibria.

However, in order to adopt Nash equilibrium as the solution concept of mechanisms, the agents must share such information as preference among them although the social planner does not have it. Actually, in many economic application, such information is private and known neither to the planner nor the other agents; For example, agents' evaluations to the object in an auction are usually considered private information. In order to obtain more useful insight about implementation, it is desirable to consider the problem in the incomplete information framework. Palfrey-Srivastava [21] and the succeeding paper by Jackson [13] defined the Bayesian implementation in the incomplete information framework by adopting Bayesian equilibrium as the solution concept instead of Nash equilibrium, and characterized it by extending Maskin’s result.

Their results gave a clue to us about how admissible the implementation of a social choice set is in the economy with incomplete information. However there has been a big criticism to the Bayesian implementation. Most of the criticism is about the assumption that the planner knows the belief or knowledge structure of the agents. In the Bayesian implementation literature, it is required that the belief and knowledge structure be commonly known among the agents and planner as usual Bayesian games. However it is unrealistic that the planner can seize such an information structure in the real economy.

It was Bergemann-Morris [6] that initially tried to answer this question. They introduced robust implementation as a belief-free implementation concept. They defined that a mechanism robustly

\[ \text{A social choice set is a natural extension of social choice correspondence to the Bayesian setting.} \]
implements a social choice function if the mechanism implements the SCF as Bayesian equilibrium in any possible belief structure, i.e. any possible type structure over the ex post payoff type space. Therefore, if he knows what this mechanism is, the planner can implement the SCF no matter what the belief structure of the agents is. Bergemann-Morris [5] and their series of paper gave a characterization of robust implementation with explicit and implicit conditions.

Though the strong result in Bergemann-Morris's work brought a break through to the implementation literature. there are still some problems. First one is that their characterization is quite hard to apply to social choice correspondences instead of social choice functions. This is because Bergemann-Morris's work is not on the line of characterization literature by Maskin, Palfrey-Srivastava, Jackson, and etc. In the previous literature, they directly characterized the implementation by equilibrium concepts using properly adjusted monotonicity of the SCC. However, in Bergemann-Morris [5], they first established the equivalence between the set of all (mixed) Bayesian equilibria in all possible Harsanyi type spaces\(^2\) over the ex post payoff type spaces and the set of interim correlated rationalizable actions\(^3\) in all possible Harsanyi type spaces. Based on this equivalence, they characterized robust implementation through characterizing the implementation by ICR. But it not easy to extend this characterization of the ICR implementation to social choice correspondence because their logic hinges on the property that each outcome in ICR is the unique best response to some belief. It is by far more difficult to implement a SCC as the set of best responses.

Secondly, their result depends on a quite simple knowledge structure. In their series of papers, they assumed that the source of uncertainty is the product space of each agent's ex post payoff type spaces, and each agent knows only his ex post payoff type but has no idea about the other agents'. However, in many economic examples, agents have some extent of information about the other agents' payoff type. For example, when the investors in the financial market receive public

\(^2\)We give a formal definition of Harsanyi type space later, but you can consider it to be a usual type structure in the Bayesian game.

\(^3\)You can see the formal argument about ICR in Dekel-Fudenberg-Morris [10].
signals such as the GDP report by the government, although they may form different evaluations on the economy, probably they will not expect that the other agents embrace an idea which is not compatible with the report according to the common sense. When the agents receive such a noisy signals about the ex post payoff types including himself, it is not enough to consider only Harsanyi type spaces in order to implement SCC or SCF’s. If the agents know the range of each other’s payoff type, such information must be incorporated into the messages to the planner. If there is a contradiction among the messages about the payoff types, it means that some agents did not report the true information to the planner. Then, the planner cannot implement the mechanism based on the false messages. Harsanyi type spaces do not include any information about the knowledge of the agents. Therefore we cannot deal with this kind of message incompatibility in Harsanyi type spaces.

One way to deal with such noisy signal models is to adopt the knowledge-belief space (hereafter abbreviated to KB space) instead of Harsanyi type spaces. KB spaces has been widely used in games with incomplete information since Aumann [2] introduced the notion. On the space, each state represents not only the agents’ beliefs but also their knowledge about the sates of the world as knowledge partitions. Palfrey-Srivastava already paid an attention to the message incompatibility problem brought by noisy signals and worked on KB spaces. As a result, although they showed Bayesian incentive compatibility (BIC) is a necessary condition for the Bayesian implementation the same as Jackson, they found generally this incentive compatibility is not sufficient for the Bayesian implementation even if we add a monotonicity condition. It makes a sharp contrast with the work by Jackson and Bergemann-Morris.

However we cannot extend Bergemann-Morris’s method directly to the implementation problem on KB spaces. They characterized robust implementation through ICR implementation. But ICR is the set of rationalizable actions based on the agents’ subjective beliefs. What the agents know does not matter there. Therefore, we take a totally different approach from theirs in order to extend Palfrey-Srivastava’s result and resolve the above two problems in Bergemann-Morris.
We begin with thinking about Bayesian implementation on the *universal type space*, shown by Mertens-Zamir [20], over the ex post payoff parameter space. According to their result, any Harsanyi type space can be embedded into the universal type space. It means that if a SCC is Bayesian implementable on the universal type space, it is robustly implementable. However the inverse direction does not necessarily holds. Friedenberg-Meier [12] named the former property *Pull-Back property* and the latter *Extension property*. They showed Bayesian equilibrium satisfies Pull-Back property, but not necessarily Extension property. This universal type space approach has another problem due to *redundant types*, which were carefully examined by Ely-Peski [11]. Their work showed that some Harsanyi types over the basic uncertainty have several different types in it which fall onto the same belief hierarchy. It implies that we cannot embed such Harsanyi type spaces into the universal type space. Bergemann-Morris’s definition of robust implementability is the implementability on any Harsanyi type space over the basic uncertainty. Therefore Bayesian implementation on the universal type space does not necessarily cover Harsany type spaces with redundancy.

For these two obstacles, we offer the following solution. To overcome the first problem, we construct the *universal knowledge-belief space* over the basic uncertainty. The universal knowledge belief space was constructed for the first time by Meier [19]. As Friedenberg-Meier discussed, whether Extension property holds for Bayesian equilibrium depends on the agents’ knowledge about the context of games. In KB spaces, we can derive the result that every agent knows they are in the states of the world. As a result, we show that each agent knows what belief (Harsanyi) type space they are in. It implies that Extension property holds in KB spaces.

However, Meier constructed it without topology by using the syntactic method a la Aumann [3]. But, as we explain soon, we require knowledge-belief spaces to be endowed with topologies in order to deal with redundant types. Once we introduce topology there, we cannot apply the syntactic

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4There is a well-known problem concerning redundancy. We discuss it later.
method any more and we have to try the semantic approach. Our strategy is to assume the all spaces including KB spaces are compact metrizable, and then apply the Kolmogorov extension theorem to construct the universal compact metrizable KB space. In order to apply the Kolomogorov extension theorem, we restrict the class of KB spaces to the one where each knowledge set in KB spaces is compact. As long as we work with compactness of the spaces and the continuity of functions, we do not think this restriction brings a serious loss of generality. We denote this universal space as the continuous universal knowledge-belief space. However, the redundant type problem still exists in the continuous universal KB space. To resolve this problem and give an epistemic foundation to redundant types, we introduce a payoff irrelevant parameter space into the basic uncertainty which was first studied by Liu [15]. In this paper, we follow Yokotani’s [23] approach. We adopt a payoff irrelevant parameter $C$ as homeomorphic to the Hilbert cube. Since the all KB spaces under consideration are compact metrizable spaces, we can embed them to compact subsets of $C$. As a result, the information attribute which cannot be captured by the belief hierarchies over the payoff parameter space can be completely represented as the hierarchies over the payoff and payoff irrelevant parameter spaces. The resulting universal continuous KB space over this augmented basic uncertainty is large enough for compact KB spaces to be embedded even if they have redundant types. Also it can be interpreted as compact KB space over the payoff parameter space.

The above facts result is the following conclusion; A social choice correspondence is Bayesian implementable in the augmented universal continuous KB space if and only if it is robustly implementable in Bergemann-Morris’s definition. As a result, we obtain a necessary and sufficient condition of robust implementation for SCC by applying Jackson’s result, and a necessary condition and a sufficient condition in the noisy signal framework.
2 Preliminaries

Throughout this paper, we assume that all spaces are endowed with some topology. We make it more specific as we go on depending on necessity.

2.1 Mathematical notations

We briefly explain mathematical symbols required for the following arguments. Let $Y$ be an arbitrary set endowed with some topology. Then we use $\Sigma(Y)$ for the Borel sets on $Y$, and $\Delta(Y)$ for the set of probability measures on $(Y, \Sigma(Y))$ endowed with $w^*$-topology. $K(Y)$ is the set of compact subspaces of $Y$ endowed with Hausdorff topology.\(^5\)

3 Robust implementation

3.1 Bayesian implementation

Let $N$ be the set of agents, $X$ be the alternative space. Let $S_i$ be the (ex post) payoff parameter space of the agent $i$. We define $S \equiv \Pi_{i \in N} S_i$. The optimal or targeted allocation for the planner is represented as a social choice correspondence (hereafter SCC) $F : S \rightharpoonup X$. If $F$ is function, we call it a social choice function. The agents’ (ex post) utility depends on alternatives and agents’ payoff types. Therefore the agent $i$’s utility is given by a function $u_i : X \times S \to \mathbb{R}_+$. It is worth while to mention that we allow the interdependent type. Then, at the interim stage, the agents form the conjecture over $S$, the other agents’ conjecture over $S$, and so on. Such hierarchies of belief and some other uncertainty make the interim states of the world, i.e. the interim type space or the Harsanyi type space. The definition is as follows;

\(^5\)See details in Kechris [14].
Definition 3.1. A space $T \equiv \langle T, (\lambda_i)_{i \in N}, (\pi_i)_{i \in N} \rangle$ is a Harsanyi type space if

$$T \equiv \Pi_{i \in N} T_i,$$

$$\lambda_i : T_i \to \Delta(T_{-i}),$$

$$\pi_i : T_i \to S_i.$$ 

Here $\pi_i$ means that the agent $i$ at the type $t_i$ knows his payoff type $\pi_i(t_i)$, and $\lambda_i$ represents the agent $i$’s conjecture over the other agents’ types. The planner has to think about how to implement a SCC $F$ at the interim stage. Palfrey-Srivastava [21] and Jackson [13] studied implementation at the interim stage, which is widely known as Bayesian implementation.

At the interim stage, we have to think about not only payoff types but also epistemic types. So SCC must be slightly modified as in Palfrey-Schmeidler [22]. The planner’s target is, instead of SCC, a social choice set (hereafter SCS). A SCS is given by a set of functions $F \subset \mathcal{F} \equiv \{ q \mid q : T \to X \}$. Bayesian implementability is about whether we can achieve $F$ as an equilibrium with some mechanism. Formally, mechanism is a family $\mathcal{M} \equiv \langle M, g \rangle$, where $M \equiv \Pi_{i \in N} M_i$ is the message space and $g : M \to X$ is the payoff function. The pair of a mechanism and a Harsanyi type space constitutes a Bayesian game $\Gamma(\mathcal{M}, T)$. Here we assume that each agent $i$ has vNM preference $R_i$, that is

$$\forall q, r \in \mathcal{F}, \quad q \ R_i(t_i) \ r \quad \text{if and only if} \quad \int_T u_i(q(t))d\lambda_i(t_i) \geq \int_T u_i(r(t))d\lambda_i(t_i).$$

The equilibrium concept is as usual. Let $G = \langle A, g \rangle$ be an arbitrary game, and $\sigma_i : T_i \to A_i$ be the agent $i$’s strategy.

Definition 3.2. (Bayesian equilibrium) The family of strategy $(\sigma_i)_{i \in N}$ is a Bayesian equilibrium

$^6$Here the mapping $\pi$ assigns a unique $s$ to each $t$, so all uncertainty about $S$ is included in the one about $T_{-i}$.
if, for each $i \in N$, $t_i \in T_i$ and $a_i \in A_i$, $\int_{T} u_i(g \circ \sigma) d\lambda_i(t_i) \geq \int_{T} u_i(g \circ (a_i, \sigma_{-i})) d\lambda_i(t_i)$.

In uncountable type space, we focus only on measurable equilibria. The next definition is from Jackson.

**Definition 3.3.** *(Bayesian implementation)* A mechanism $\mathcal{M}$ Bayesian implements a SCS $F$ on a Harsanyi type space $T$ if, (1) for each $f \in F$, there exists a Bayesian equilibrium $\sigma$ in $\Gamma$ such that $f = g \circ \sigma$ and, (2) for each equilibrium $\sigma$, there exists $f \in F$ such that $g \circ \sigma = f$.

### 3.2 Robust implementation

However, we face a serious practical difficulty in Bayesian implementation argument. There the planner must know the Harsanyi type space of the agents in order to design a mechanism. It is not likely to be met in applications. Therefore we are required to think about some kind of “belief-free” implementation concept. Bergemann and Morris first formally defined such *robust implementation* and gave a characterization. For the rest of this section, we only deal with SCF. We discuss why it is later.

**Definition 3.4.** *(Bergemann-Morris [6])* A mechanism $\mathcal{M}$ robustly implements a SCF $f$ if $\mathcal{M}$ Bayesian implements $f$ in every Harsanyi type space $T$.

We say that a SCF $f$ is *robustly implementable* if there exists a mechanism that robustly implements $f$.

We introduce Bergemann-Morris’s characterization result without proof. They gave a necessary and sufficient condition of robust implementability using the following concepts.
Definition 3.5. *(EPIC)* A social choice function \( f \) satisfies ex-post incentive compatibility (EPIC) if, for each \( i \in N \), \( s_i, s'_i \in S_i \), and \( s_{-i} \in S_{-i} \), \( u_i(f(s_i, s_{-i}), s_i, s_{-i}) \geq u_i(f(s'_i, s_{-i}), s_i, s_{-i}) \).

Definition 3.6. *(Robust monotonicity)*

Let \( Y_i(s_{-i}) \equiv \{ y \in X | \forall s'_i \in S_i, u(y, s'_i, s_{-i}) \leq u(f(s'_i, s_{-i}), s'_i, s_{-i}) \} \). Then, a SCF \( f \) satisfies robust monotonicity if, for each unacceptable deception \( \beta \),

\[
\exists i \in N, s_i \in S_i, \text{ and } s'_i \in \beta_i(s_i) \text{ s.t.}
\forall s'_{-i} \in S_{-i}, \exists y \in Y_i(s'_{-i}) \text{ s.t.}
\forall s_{-i} \in S_{-i} \text{ with } s'_{-i} \in \beta_{-i}(s'_{-i}),
\quad u_i(y, s_i, s'_{-i}) > u_i(f(s'_i, s'_{-i}), s_i, s'_{-i}).
\]

Definition 3.7. *(Bad outcome condition)* For each \( i \in N \), there exists \( y^*_i \in X \) such that

\[
\forall s_i \in S_i, \forall s_{-i} \in S_{-i}, \text{ and } \forall \psi_i \in \Delta(S_{-i}),
\quad \text{there exists } y_i \in Y_i(s_{-i}) \text{ s.t.}
\quad \int_{S_{-i}} u_i(y_i, s_i, s'_{-i})d\psi > \int_{S_{-i}} u_i(y^*_i, s_i, s'_{-i})d\psi.
\]

They proved the next theorem.

Theorem 3.8. *(Bergemann-Morris)* Under the bad outcome condition, a SCF \( f \) is robustly implementable if and only if it satisfies EPIC and robust monotonicity.

4 The universal type space

Bergemann-Morris gave a necessary condition and a sufficient condition for robust implementation of a SCF. However they derived it through another implementation concept using rationalizable actions. Therefore their canonical mechanism that implements a SCF is different from that in the
existing implementation literature using some equilibrium concept such as Nash equilibrium and Bayesian equilibrium. One problem here is that we cannot apply the mechanism to SCC. In this paper, we take a different approach from Bergemann-Morris. We try to characterize the robust implementation by applying the existing results about Bayesian implementation to the universal type space founded by Mertens-Zamir [20]. In this section, we introduce the construction of the universal type space by Brandenburger-Dekel [8]. We assume that $S$ is a compact metrizable space and $N = \{1, 2\}$. We can extend it to a larger set of agents.

The universal type space is the space of coherent hierarchies of belief over $S$. Hierarchy of belief is the infinite sequence of the agents’ belief over $S$ and the belief over the other agents’ belief over $S$ and so on.

$$
Z^0 \equiv S,
Z^1 \equiv Z^0 \times \Delta(S),
\vdots
Z^k \equiv Z^{k-1} \times \Delta(Z^{k-1}),
\vdots
$$

Let $Z^\infty \equiv \Pi_{k \geq 0} \Delta(Z^k)$, and for each $k$, $\mu_k \in \Delta(Z^k)$. Then $Z^\infty$ is the entire set of hierarchies of belief over $S$. But we require each order of belief to be coherent to the previous order of belief. We define the coherent subset of $Z^\infty$ as

$$
T^1 \equiv \{(\mu_k)_{k \geq 1} \in Z^\infty \mid \forall k \geq 1, \ Marg(\Delta(Z^k)) \mu_{k+1} = \mu_k\}.
$$

Brandenburger-Dekel showed that there exists a homeomorphism $h : T^1 \to \Delta(S \times Z^\infty)$. However, for rational agents, $T^1$ is not enough to represent coherency because this coherency of belief must
be common knowledge between the agents. So we define the following family of subsets of $\mathbb{Z}^\infty$.

$$T^k \equiv \{ \mu \in T^{k-1} \mid h(\mu)[S \times T^{k-1}] = 1 \},$$

$$\vdots$$

$$T^* \equiv \bigcap_{k \geq 1} T^k.$$

By construction, $T^*$ is the set of coherent belief hierarchies where coherency is commonly known.

**Theorem 4.1.** (Brandenburger-Dekel [8]) There exists a homeomorphism between $T^*$ and $\Delta(S \times T^*)$.

From this theorem, we can identify this space as the universal type space by taking $U(S) \equiv \langle S, T^* \times T^*, h_1^*, h_2^* \rangle$ where $h_i^*$ is homeomorphism. And we have the theorem by Mertens-Zamir.

**Theorem 4.2.** (Mertens-Zamir [20]) Every Harsanyi type space without redundant types can be homeomorphically embedded to $U(S)$.

We formally discuss redundant types later. Then, based on this fact, we tends to think that, except for the problem with redundancy, if a SCC $F$ is Bayesian implementable on $U(S)$, it is robustly implementable. However it is not true due to the lack of Extension property of Bayesian equilibrium. We discuss it in the next section.

## 5 Extension property

As Mertens-Zamir showed, we can consider any Harsanyi type space over $S$ as a subspace of $U(S)$ unless it does not have redundant types. So, given a game $\Gamma$, we tend to think that it is sufficient to
analyze Bayesian equilibrium on $U(S)$ whatever the true type space is. It seems true that we can
deal with all Bayesian equilibria on every type space just by taking a part of Bayesian equilibria
on $U(S)$. However it turns out to be wrong as Friedenberg-Meier [12] discussed. They showed that
a part of any Bayesian equilibrium on $U(S)$ focusing on its sub type space becomes a Bayesian
equilibrium on that sub space, but not all Bayesian equilibria of a sub type space can be realized as
such a part of an equilibrium on $U(S)$. They named the former property of Bayesian equilibrium
$Pull-Back$ $property$ and the latter $Extension$ $property$. Failure of Extension property is a problem
here.

We give the formal definition of Extension property and an example of its failure by Friedenberg-
Meier. Let a game $\Gamma$ be a family of an action space $A = \Pi_{i \in N} A_i$ and a payoff function $g : A \rightarrow X$
i.e. $\Gamma \equiv \langle A, g \rangle$.

**Definition 5.1.** (Friedenberg-Meier [12]) Let $\mathcal{T}$ and $\tilde{T}$ be arbitrary Harsanyi type spaces such that
$\tilde{T} \subset \mathcal{T}$. Bayesian equilibrium satisfies the extension property if, for such $\mathcal{T}$ and $\tilde{T}$, and for each
Bayesian equilibrium $\tilde{\sigma}$ of the game $\langle \tilde{T}, \Gamma \rangle$, there exists a Bayesian equilibrium $\sigma$ of $\langle \mathcal{T}, \Gamma \rangle$ such
that $\sigma$ is an extension of $\tilde{\sigma}$ to $\langle \mathcal{T}, \Gamma \rangle$.

However this property does not necessarily hold as in the example below.

**Example (Friedenberg-Meier 2008):** Let $N = \{1, 2\}$ and consider two following type spaces.
The first one is $\mathcal{T}_0 \equiv \langle \{s^*\}, \{t^*_i\}_{i \in N}, (\lambda_i)_{i \in N} \rangle$, where for each $i \in N$, $\lambda_i(t^*_i)(s^*, t^*_{-i}) = 1$. To define
another type space, we need a preliminary type space $\mathcal{T}' = \langle S_1 \times S_2, T_1 \times T_2, (\lambda'_i)_{i \in N} \rangle$. Here, for
each $i \in N$, we assume $s^*_i \notin S_i$ and $t^*_i \notin T_i$. The second one is a kind of compound type space of $\mathcal{T}_0$.
and $T'$.

$$T_1 \equiv (S \cup \{s^\ast\}, (T_i \cup \{t_i^\ast\})_{i \in N}, (\lambda_i)_{i \in N})$$ such that

$$\forall i \in N, t_i \in T_i, \lambda_i(t_i)[(s^\ast, t_{i-1}^\ast)] = p,$$

$$\lambda_i(t_i)[E] = (1 - p)\lambda'_i(t_i)[E]$$ for each $E \subset [S \times T_{i-1}]$.

The belief structure of $T_1$ is shown in the picture below.

From this picture, it is clear that $T_0$ is represented as a point $(s^\ast, t_1^\ast, t_2^\ast)$ in $T_1$.

Next we define a game on these type spaces. Let $\Gamma = ((A_i \cup \{a_i^\ast\})_{i \in N}, g)$, where $g : S \cup \{s^\ast\} \times A \cup \{a^\ast\} \rightarrow \mathbb{R}^2$. The detailed structure of $g$ is given in the next picture.
Player 1’s payoff at $s \in S$

<table>
<thead>
<tr>
<th>$a_1^*$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g_1(a_1, a_2)$</td>
<td></td>
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</table>

Player 1’s payoff at $s^*$

<table>
<thead>
<tr>
<th>$a_1^*$</th>
<th>0</th>
<th>0</th>
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<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_2^*$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here $x, y > 0$, and for each $(a_1, a_2) \in A_1 \times A_2$, $g_1(a_1, a_2) \in [1, 2]$. The agent 2’s payoff is just symmetric. And we have another assumption about $g$.

**Assumption 5.2.** The Bayesian game $G' \equiv \langle T', \Gamma' = (A, g) \rangle$ is discontinuous, and does not have Bayesian equilibrium.

Next theorem shows Extension property does not hold for the two games $G_0 \equiv \langle T_0, \Gamma \rangle$ and $G_1 \equiv \langle T_1, \Gamma \rangle$.

**Theorem 5.3.** *(Friedenberg-Meier[12])* Extension property does not necessarily holds.

(Sketch of proof) You can see detailed proof in their paper, so we only give the sketch of proof. There is a Bayesian equilibrium $\sigma^0$ of $G_0$ such that, for each $i \in N$, $\sigma^0_i(t^*_i) = a_i \in A_i$. If Extension property holds, there exists a Bayesian equilibrium $\sigma^1$ of $G_1$ such that, for each $i \in N$, $\sigma^1_i(t^*_i) = a_i \in A_i$. Then, is clear that $\sigma^1_i(t^*_i) \in A_2$. Then, from the figure, we know that $a_1^*$ is strictly dominated at each $t_1 \in T_1$, which means that $\sigma^1(t_i) \in A_1$ for all $t_1 \in T_1 \cup \{t^*_i\}$. By the same logic, $a_2^*$ is strictly dominated at each $t_2 \in T_2$. Thus the equilibrium strategy $\sigma^1$ only takes actions only

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7By choosing proper $g$, we can satisfies this condition. See details in Friedenberg-Meier and Sion-Wolf
from $A$, so by focusing on the subset $(S_1 \times S_2, T_1 \times T_2)$, $\sigma^1$ is also strategy on the game $G'$. From the belief structure in the figure, we know that $\sigma^1$ is a Bayesian equilibrium on the game $G'$. It is contradiction. □

6 The universal knowledge-belief space

6.1 Knowledge-belief space

We discussed the failure of the Extension property on the universal type space. It happens because the agents do not know the "context" of the game. Here 'context' means the true type structure they belong to. If the agents know the context at each epistemic type, it is possible for this problem to be resolved. For this purpose, we introduce "knowledge" in addition to subjective belief. We adopt knowledge-belief (hereafter KB space) space, which was introduced by Aumann [4] and others. The KB space is the Harsanyi type space with the knowledge partition by Aumann [3].

**Definition 6.1.** A tuple $V \equiv (V \subset S \times T, (\lambda_i)_{i \in N}, (P_i)_{i \in N})$ is a knowledge-belief space if it satisfies

1. $\lambda_i : T_i \rightarrow \Delta(S \times T_{-i})$,
2. $P_i : T_i \rightarrow \mathcal{K}(S \times T_{-i})$,
3. $\forall i \in N, \ t_i \in T_i, \ \text{Supp}(\lambda_i(t_i)) \subset P_i(t_i)$,
4. $\forall i \in N, \ \forall t_i, \ \exists ! s_i \in S_i \ s.t. \exists s_{-i} \in S_{-i} \ and \ s \in P_i(t_i)$,
5. $\forall i \in N, \ t_i \in T_i$,
   
   if $(s, t_i, t_{-i}) \in P_i(t_i)$, then, $\forall j \in N, \ (s, t_i, t_{-i}) \in P_j(t_j)$,
6. $V \subset S \times T \ s.t. \ V = \{(s, t) \in S \times T \ | \ \forall i \in N, \ (s, t) \in P_i(t_i)\}$. 

16
Note that any Harsanyi type space can be interpreted as a KB space where every agent’s knowledge partition consists of the entire space. Therefore we can consider the whole domain of KB spaces includes all Harsanyi type spaces.

From (2) in Definition 6.1, it is derived that every agent knows the context. The next theorem shows that the Extension property is satisfied on KB spaces. First we show that every KB space is decomposable.

**Lemma 6.2.** Let $\mathcal{V} \equiv \langle V, (\lambda_i)_{i \in N}, (P_i)_{i \in N} \rangle$ be a knowledge-belief space, and $\mathcal{V}_0 \equiv \langle V_0, (\lambda_i)_{i \in N}, (P_i)_{i \in N} \rangle$ be its sub type space. Then, the complement $\langle V_c^0, (\lambda_i)_{i \in N}, (P_i)_{i \in N} \rangle$ becomes a knowledge-belief space.

**Proof.** Let $x \in V_c^0$ be an arbitrary point in the complement of $V$. Suppose that, for some $i \in N$, $P_i(x_i) \cap V_0 \neq \emptyset$. Let $y \in P_i(x_i) \cap V_0$. By the definition, for each $i \in N$, $y \in P_i(y_i)$ and $P_i(y_i) \subset V_0$. However, the condition (5) in the definition 6.1 implies that $x \in P_i(y_i)$. It means that $P_i(y_i) \cap V_c^0 \neq \emptyset$. It contradicts the fact that $P_i(y_i) \subset V_0$. Therefore $P_i(x_i) \subset V_c^0$. Since $\mathcal{V}$ is a K-B space, the conditions (1)-(6) in 6.1 are satisfied at $x$. So the fact $P_i(x_i) \subset V_c^0$ implies that the conditions (1)-(6) are satisfied for $\langle V_c^0, (\lambda_i)_{i \in N}, (P_i)_{i \in N} \rangle$. □

As a corollary of Lemma 6.2, we have the next theorem.

**Theorem 6.3.** As long as there exists one Bayesian equilibrium on the space, the Extension property for Bayesian equilibrium holds in any KB space.
6.2 The universal knowledge-belief space

Since we showed that Extension property of Bayesian equilibrium holds in KB spaces, the next thing to do is find the “universal” KB space. Meier [19] showed that the existence of the universal KB space in the case without topology by using syntactical method. However we want to find it in the topological case which are assumed in the most of the literature. For the topological argument, we define the continuous knowledge-belief space. In the rest of the paper, we focus on this class of KB spaces.

Definition 6.4. A KB space $\mathcal{V}$ is a continuous knowledge-belief space if, for each $i \in \mathbb{N}$, $S_i$, $T_i$ and $V$ are compact metrizable spaces and, for each $i \in \mathbb{N}$, $\lambda_i$ and $P_i$ are continuous.

To construct the universal continuous KB space, we construct the space of hierarchies of knowledge and belief over $S$ in a similar way as we did in the section 3.

$$
\tilde{Z}_0 \equiv S,
\tilde{Z}_1 \equiv \tilde{Z}_0 \times \Delta(\tilde{Z}_0) \times \mathcal{K}(\tilde{Z}_0),
\vdots
\tilde{Z}_k \equiv \tilde{Z}_{k-1} \times \Delta(\tilde{Z}_{k-1}) \times \mathcal{K}(\tilde{Z}_{k-1}),
\vdots
$$

Let $T_0 \equiv \Pi_{k \geq 0}\{\Delta(\tilde{Z}_k) \times \mathcal{K}(\tilde{Z}_k)\}$. We focus on the coherent hierarchies of belief and knowledge in

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8Mertens-Zamir, Brandenburger-Dekel, etc
Let $\tau_n \equiv (\mu_n, \kappa_n) \in \Delta(\tilde{Z}_{n-1}) \times \mathcal{K}(\tilde{Z}_{n-1})$, and

$$T_1 \equiv \{ (\tau_n)_{n \geq 1} \in T_0 \mid \forall n \geq 1, \text{Proj}_{\Delta(\tilde{Z}_n) \times \mathcal{K}(\tilde{Z}_n)} \mu_{n+1} = \mu_n, \ \text{and} \ \text{Proj}_{\Delta(\tilde{Z}_n) \times \mathcal{K}(\tilde{Z}_n)} \kappa_{n+1} = \kappa_n \}.$$

As Brandenburger-Dekel [8] and Mariotti et al. [17], we can apply Kolmogorov extension theorem to obtain the next result.

**Lemma 6.5.** Let $\{Z_k\}_{k \geq 0}$ be a countable family of compact metrizable spaces. Let $Z^n \equiv \Pi_{0 \leq k \leq n} Z_k$ and $Z^\infty \equiv \Pi_{k \geq 0} Z_k$. Let

$$T \equiv \{ (\tau_n)_{n \geq 1} \mid \forall n, \ \tau_n = (\mu_n, \kappa_n) \ \text{s.t.} \ \begin{align*} & (1) \ \mu_n \in \Delta(Z^{n-1}), \ \kappa_n \in \mathcal{K}(Z^{n-1}), \ \text{and} \ \text{Proj}_{\Delta(Z_{n-1})} \mu_{n+1} = \mu_n, \ \text{Proj}_{\Delta(Z_{n-1})} \kappa_{n+1} = \kappa_n \}. \end{align*}$$

Then there exists homeomorphism $h : T \to \Delta(Z^\infty) \times \mathcal{K}(Z^\infty)$ such that, for each $n \geq 1$, $\text{Proj}_{Z^n} h((\tau_k)_{k \geq 1}) = (\mu_n, \kappa_n)$.

**Proof.** Given $(\tau_n)_{n \geq 1} \in T$, let $\tilde{\mu}$ be the Kolmogorov extension of $(\mu_n)_{n \geq 1}$ to $Z^n$. We can also define compact spaces such that $K_n \equiv \kappa_n \times \Pi_{k \geq n} Z_k$ and $K \equiv \bigcap_{n \geq 1} K_n$.

Let $h : T \to \Delta(Z^\infty) \times \mathcal{K}(Z^\infty)$ be such that $(\tau_n)_{n \geq 1} \mapsto (\tilde{\mu}, K)$. By construction, we can say that $h$ is injection. And for each $(\mu, \kappa) \in \Delta(Z^\infty) \times \mathcal{K}(Z^\infty)$, we can find a sequence $(\tau_n)_{n \geq 1} = (\mu_n, \kappa_n)_{n \geq 1}$ such that $\text{Proj}_{Z^n} (\mu, \kappa) = (\mu_n, \kappa_n)$ by taking projection over $Z^n$ of $\mu$ and $\kappa$ respectively. From these results, we have that $h$ is bijection. Since, for each $k$, $Z_k$ is compact metrizable, the mapping $h_\Delta : (\mu_n)_{n \geq 1} \mapsto \mu$ and $h_\mathcal{K} : (\kappa_n)_{n \geq 1} \mapsto \kappa$ are continuous mapping. Therefore $h$ is continuous bijection.

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9See Brandenburger-Dekel [8] and Mariotti et al [17].
tion from $T$ to $\Delta(Z^\infty) \times K(Z^\infty)$. Both $T$ and $\Delta(Z^\infty) \times K(Z^\infty)$ are compact metrizable spaces. Thus $h$ is homeomorphism. □

We have the next proposition as a Corollary of Lemma 6.5.

**Proposition 6.6.** The set of coherent knowledge-belief hierarchies $T_1$ is homeomorphic to $\Delta(S \times T_0) \times K(S \times T_0)$.

Let $T^*$ be the set of coherent KB hierarchies that satisfy the common knowledge of coherency, i.e.,

$$T_k \equiv \{ \tau \in T_{k-1} \mid h_1(\tau)[S \times T_{k-1}] = 1, \text{ and } h_2(\tau) \in K(S \times T_{k-1}) \},$$

=: $$T^* \equiv \bigcap_{k \geq 1} T_k.$$

We show that $T^*$ is homeomorphic to $\Delta(S \times T^*) \times K(S \times T^*)$.

**Proposition 6.7.** The set of coherent knowledge-belief hierarchies $T^*$ is non-empty, and homeomorphic to the space $\Delta(S \times T^*) \times K(S \times T^*)$.

**Proof.** Let $h$ be a homeomorphism from $T_1 \rightarrow \Delta(S \times T_0) \times K(S \times T_0)$. We define

$$\tilde{T}^* \equiv \{ \tau \in T_1 \mid h_1(\tau)[S \times T^*] = 1, \text{ and } h_2(\tau) \in K(S \times T^*) \}.$$ 

By construction, for each $\tau \in T^*$, $h_1(\tau)[S \times T^*] = 1$, and $h_2(\tau) \in K(S \times T^*)$. Therefore, $T^* \subset \tilde{T}^*$. On the other hand, for each $\tau \in \tilde{T}^*$, $h_1(\tau)[S \times T_1] = 1$, and $h_2(\tau) \in K(S \times T_1)$. It means that $\tilde{T}^* \subset T_1$. Recursively, we have that, for each $n \geq 1$, $\tilde{T}^* \subset T_n$. Therefore $\tilde{T}^* \subset T^*$, which concludes that $\tilde{T}^* = T^*$.  

20
From the above results,

\[ h(T^*) \equiv \{ \tau = (\mu, \kappa) \in \Delta(S \times T_0) \times \mathcal{K}(S \times T_0) \mid \mu[S \times T^*] = 1, \text{ and } \kappa \in \mathcal{K}(S \times T^*) \}. \]

Since \( T^* \) is a compact subset of \( T_0 \), \( \Delta(S \times T^*) \) is homeomorphic to the space \( \{ \mu \in \Delta(S \times T_0) \mid \mu[S \times T^*] = 1 \} \) and \( \mathcal{K}(S \times T^*) \) is homeomorphic to \( \{ \kappa \in \mathcal{K}(S \times T_0) \mid \kappa \in \mathcal{K}(S \times T^*) \} \). Let \( g_1 \) and \( g_2 \) be the homeomorphism respectively. Then \( g \equiv (g_1, g_2) \) is homeomorphism from \( \Delta(S \times T^*) \times \mathcal{K}(S \times T^*) \) to \( h(T^*) \). It concludes that the mapping \( g^{-1} \circ h : T^* \to \Delta(S \times T^*) \times \mathcal{K}(S \times T^*) \) is homeomorphism. □

Let \( U^*(S) \equiv S \times \Pi_{i \in N} T^i \). Hereafter we also use the terms \( U^*(S) \) and its subspaces to mean the KB structures associated with the homeomorphism \( \lambda^* \) and \( P \).

As in the universal type space, we show that any continuous KB space without a certain class, the KB spaces with redundant types, can be embedded to \( U^* \). Before that we clarify what is it that two KB spaces are homeomorphic.

**Definition 6.8.** \((S\text{-homeomorphism})\) Let \( \mathcal{V} \) and \( \mathcal{W} \) be continuous KB spaces. A mapping \( h : V \to W \) is \( S\)-homeomorphism if

1. \( h \) is homeomorphism,
2. \( \forall i \in N, \forall t_i \in T_i, \quad \lambda_i(t_i) \circ h^{-i} = \phi_i(h(t_i)), \)
3. \( \forall i \in N, \forall t_i \in T_i, \quad h[P_i(t_i)] = \bar{P}_i(h(t_i)), \)

where \( v \equiv (t_i, t_{-i}, s) \in V \), and \( h(t_i) \equiv \text{Proj}_{i(T_i)} h(v) \).

First we show the next lemma.

**Lemma 6.9.** Let \( \mathcal{V} \) be a continuous KB space. Then there exists a continuous mapping \( \gamma : V \to \)
\[ U^*(S). \]

**Proof.** Given types in continuous KB spaces, we can derive the hierarchy mapping \( \gamma \) from types to their sequential knowledge-belief over \( S \) as follows;

\[ \forall i \in N, \quad \gamma^1_i : t_i \mapsto (\text{Projs}\lambda_i(t_i), \text{Projs}P_i(t_i)), \]

\[ \gamma^2_i : t_i \mapsto [(\text{Projs}\lambda_i(t_i), \text{Projs}P_i(t_i)), (\lambda_i(t_i) \circ (\gamma^1_{i-1})^{-1}, \gamma^1_{i-1}[P_i(t_i)])], \]

\[ \vdots \]

\[ \gamma^k_i : t_i \mapsto [\cdots, (\lambda_i(t_i) \circ (\gamma^{k-1}_{i-1})^{-1}, \gamma^{k-1}_{i-1}[P_i(t_i)])], \]

\[ \vdots \]

Let \( \gamma^\infty_i \equiv \lim_{k \to -\infty}(\gamma^k_i) \). We show that this inverse limit \( \gamma^\infty_i \) is a continuous mapping from \( V \) to \( \Omega \).

Let \( (t_i^l)_{l \in N} \) be such that, for each \( l \in N \), \( t_i^l \in T_i \) and there exists \( t_i^* \in T_i \) with \( t_i^* = \lim_{l \to -\infty} t_i^l \). We want to show that, for each level \( k \in N \), \( \lim_{l \to -\infty} \gamma^k_i(t_i^l) = \gamma^k_i(t_i^*) \).

We use induction. For each \( i \in N \), at \( k = 1 \), we have \( \gamma^1_i(t_i^*) = \lim_{l \to -\infty} \gamma^1_i(t_i^l) \) because \( \lambda_i \) and \( P_i \) are both continuous. Suppose that, for each \( i \in N \), \( \gamma^k_i \) is continuous. Then, \( \lambda_i(t_i^l) \circ (\gamma^{k-1}_{i-1})^{-1} \to \lambda_i(t_i^*) \circ (\gamma^{k-1}_{i-1})^{-1} \) as \( l \to \infty \) because \( \lambda_i \) is continuous and \( \gamma^k_{i-1} \) is fixed. Also since \( \gamma^k_{i-1} \) is continuous, we have \( \gamma^k_{i-1}[P_i(t_i^l)] \to \gamma^k_{i-1}[P_i(t_i^*)] \) as \( l \to \infty \). It means that \( \gamma^{k+1}_i(t_i^*) = \lim_{l \to -\infty} \gamma^{k+1}_i(t_i^l) \).

Therefore, for each \( k \in N \), \( \lim_{l \to -\infty} \gamma^k_i(t_i^l) = \gamma^k_i(t_i^*) \), which means that \( \gamma^\infty_i \) is continuous. \( \square \)

**Definition 6.10.** Let \( t_i \) and \( t_i' \) be types a continuous KB space \( V \). The types \( t_i \) and \( t_i' \) are redundant types if they are mapped to the same hierarchy by the hierarchy mapping \( \gamma \).
**Proposition 6.11.** As long as a continuous KB space $\mathcal{V}$ does not have redundant types, the hierarchy mapping $\gamma : V \rightarrow U^*(S)$ is $S$-isomorphism.

**Proof.** Since $\mathcal{V}$ does not have redundant types, the hierarchy mapping $\gamma_i^\infty$ is bijection. Therefore, by Theorem 2.36 in Aliprantis-Border [1], $\gamma_i^\infty$ is (homeomorphic) embedding from $V$ to $\Omega$. □

Next we consider a subspace of $U^*(S)$ as follows:

$$\Omega(S) \equiv \{ \omega \in U^*(S) \mid \text{There exists a continuous KB space } \mathcal{V} \text{ with } \omega \in \mathcal{V} \}$$

Our plan is to show that $\Omega(S)$ is the *universal continuous metrizable KB space*, in the sense that we can embedd any continuous KB space there. To do that, it is sufficient to show that $\Omega(S)$ is a continuous KB space.

For this purpose, we define another subspace of $U^*(S)$ similar to $\Omega(S)$;

$$\tilde{\Omega}(S) \equiv \{ \omega \in U^*(S) \mid \text{There exists a KB space } L \text{ with } \omega \in L \}$$

Note that $\Omega(S) \subset \tilde{\Omega}(S)$ from the construction.

Now let us a continuous KB space without the compactness of spaces be a *semi-continuous KB space*. At the next step, we show $\tilde{\Omega}$ is a semi-continuous KB space.

**Lemma 6.12.** The space $\tilde{\Omega}(S)$ is a semi-continuous KB space.

**Proof.** As shown above, there exist the homeomorphism $\lambda^*_i : T^* \rightarrow \Delta(S \times T^*)$ and $P_i : T^* \rightarrow$
$K(S \times T^*)$. By restricting the domain, we obtain the continuous functions $\lambda_i^* : Proj_T^* \tilde{\Omega}(S) \to \Delta(ProjS \times T^* \tilde{\Omega}(S))$ and $P_i : Proj_T^* \tilde{\Omega}(S) \to K(ProjS \times T^* \tilde{\Omega}(S))$. Then, from (3) in Definition 6.1, we have, for each $\omega \in \tilde{\Omega}(S)$, $\text{Supp}(\lambda_i^*(\omega)) \subset P_i^*(\omega)$. And each $\omega$ is in some KB subspace $L$, so (4) and (5) in Definition 6.1 are satisfied. For each $\omega \in \tilde{\Omega}(S)$, there exists a KB space $L$ with $\omega \in L$. Therefore, for each $i \in N$, $\omega \in P_i(\omega_i)$. It means $(\tilde{\Omega}(S), \lambda^*, P^*)$ is a semi-continuous KB space. \qed

Next we show that $\tilde{\Omega}(S)$ is a continuous KB space by showing that $\tilde{\Omega}(S)$ is compact.

**Proposition 6.13.** The space $\tilde{\Omega}(S)$ is a compact space.

**Proof.** Since $\tilde{\Omega}(S) \subset U^*(S)$ and $U^*(S)$ is compact, we only have to show that $\tilde{\Omega}(S)$ is closed. Let $(\omega^k)_{k \in \mathbb{N}}$ be the sequence such that, for each $k \in \mathbb{N}$, $\omega^k \in \tilde{\Omega}(S)$ and there exists a limit $\omega^* \in U^*(S)$ i.e. $\lim_{k \to \infty} \omega^k = \omega^*$. From the lemma above, $\tilde{\Omega}(S)$ is a semi-continuous KB space. Therefore, for each $k \in \mathbb{N}$ and $i \in N$, it holds that $\text{Supp}(\lambda_i^*(\omega^k)) \subset P_i^*(\omega^k)$. And it also implies that, for each $i \in N$ and $k \in \mathbb{N}$, $\lambda^*(\omega^k)[P_i^*(\omega^k)] = 1$. Remind that $\lim_{k \to \infty} \omega^k = \omega^*$ by the assumption. By the continuity of $P_i^*$, we have, for each $i \in N$, $P_i^*(\omega_i^k) \to P_i^*(\omega_i^*)$ in Vietoris topology.

So, for each $\epsilon > 0$, there exists $k_1$ such that, for each $k \geq k_1$, $H_d(P_i^*(\omega^k), P_i^*(\omega^*)) \leq \epsilon_1$. Therefore, for each $k \geq k_1$, $P_i^*(\omega^k) \subset B_{cl}(P_i^*(\omega^*), \epsilon_1)$. Here $B_{cl}(X, \epsilon)$ stands for the $\epsilon$-closed ball of the set $X$. It means that $\lambda_i^*(\omega^k)[B_{cl}(P_i^*(\omega^*), \epsilon_1)] = 1$. By the construction, $\lambda_i^*(\omega^k) \to \lambda_i^*(\omega^*)$ as $k \to \infty$. So we have $\lim_{k \to \infty} \lambda_i^*(\omega^k)[B_{cl}(P_i^*(\omega^*), \epsilon_1)] = \lambda_i^*(\omega^*)[B_{cl}(P_i^*(\omega^*), \epsilon_1)]$, which means that $\lambda_i^*(\omega^*)[B_{cl}(P_i^*(\omega^*), \epsilon_1)] = 1$. In the same way, for each $0 < \epsilon \leq \epsilon_1$, $\lambda_i^*(\omega^*)[B_{cl}(P_i^*(\omega^*), \epsilon_1)] = 1$. We can pick up a sequence $(\epsilon_i)_{i=1}^{\infty}$ such that $\lim_{i \to \infty} \epsilon_i = 0$. Then, since $(B_{cl}(P_i^*(\omega^*), \epsilon_i))_{i=1}^{\infty} \subseteq \mathbb{R}$ is monotonically decreasing, $\lim_{i \to \infty} B_{cl}(P_i^*(\omega^*), \epsilon_i) = P_i^*(\omega^*)$. Since, for each $l$, $\lambda_i^*(\omega^*)[B_{cl}(P_i^*(\omega^*), \epsilon_i)] = 1$, $\lambda_i^*(\omega^*)[P_i^*(\omega^*)] = 1$. So $\text{Supp}(\lambda_i^*(\omega^*)) \subset P_i^*(\omega^*)$. The convergence of the knowledge sets $(P_i^k)_{k \in \mathbb{N}}$ also implies that, for each $i \in N$ and $\epsilon > 0$, $P_i^*(t^*_i) \subset B(\tilde{\Omega}(S), \epsilon)$.
Before going further, we consider the condition (4) in the Definition 6.1. We can easily check that there exists $s_i^* \in S_i$ such that $\text{Proj}_{(S_i)}P_i(t_i^*) = \{s_i^*\}$ because $s_i^* = \lim_{k \to \infty} s_i^k$ with $\text{Proj}_{(S_i)}P_i(t_i^k) = \{s_i^k\}$.

Let $\bar{\Omega}(S)$ be the closure of $\tilde{\Omega}(S)$. For each $\omega \in \bar{\Omega}(S) \backslash \tilde{\Omega}(S)$, it is the limit of a convergent sequence in $\bar{\Omega}(S)$. Therefore, from the preceding results, for each $i \in N$, $\text{Supp}(\lambda_i^*(\omega_i)) \subset P_i(\omega_i)$. From the above argument, we have, for any $\epsilon > 0$, $P_i(\omega_i) \subset B(\bar{\Omega}(S), \epsilon)$. So, for any $\epsilon > 0$, it holds that $P_i(\omega_i) \subset B(\bar{\Omega}(S), \epsilon)$. Since the set $\bar{\Omega}(S)$ is closed, $P_i(\omega_i) \subset \bar{\Omega}(S)$. As a result, $\bar{\Omega}(S)$ satisfies Definition 6.1 except for (5) and (6).

Finally we show that (5) and (6) in Definition 6.1 holds at $\omega \in \bar{\Omega}(S)$. Suppose that $\omega \in \bar{\Omega}(S) \backslash \tilde{\Omega}(S)$. Otherwise it is clear that (5) and (6) are satisfied. Let $\tilde{\omega} \in P_i(\omega)$. All we have to show is (1) For all $j \in N$, $\omega \in P_i(\omega)$, and (2) For all $j \neq i$, $\tilde{\omega} \in P_j(\tilde{\omega})^{10}$. Now there exists $(\omega^k)_{k \in \mathbb{N}}$ such that, for each $k$ and $j \in N$, $\omega^k \in P_j(\omega^k)$ and $\lim_{k \to \infty} \omega^k = \omega$. By the continuity of $P_j$, for each $j \in N$, $\lim_{k \to \infty} \omega^k \in \lim_{k \to \infty} P_j(\omega^k)$, that is, $\omega \in P_j(\omega)$.

Next we show that there exists a subsequence $(\omega_{k_n})_{k_n \in \mathbb{N}}$ such that there exists $(\tilde{\omega}_n)_{n \in \mathbb{N}}$ with, for each $n$, $\tilde{\omega}_n \in P_i(\omega_{k_n})$ and $\tilde{\omega}_n \to \tilde{\omega}$ as $n \to \infty$. We can construct such a sequence in the following way. By the definition of the Hausdorff topology, for each $\delta_n > 0$, we have $d(\tilde{\omega}, P_i(\omega_{k_n})) < \delta_n$ for any sufficiently large $n$. Since $P_i(\omega_{k_n})$ is compact and the distance function is continuous, there exists $\tilde{\omega}_n \in P_i(\omega_{k_n})$ such that $d(\tilde{\omega}, \tilde{\omega}_n) < \delta_n$. We can find such $\tilde{\omega}_n$ for each $\delta_n > 0$. Clearly, the sequence $(\tilde{\omega}_n)_{n \in \mathbb{N}}$ converges to $\tilde{\omega}$. Since, for each $k_n$, $\omega_{k_n} \in \tilde{\Omega}(S)$, for each $j \in N$, $\tilde{\omega}_n \in P_j(\tilde{\omega}_n)$. So the continuity of $P_j$ implies that $\lim_{n \to \infty} \tilde{\omega}_n \in \lim_{n \to \infty} P_j(\tilde{\omega}_n)$, that is, $\tilde{\omega} \in P_j(\tilde{\omega})$.

The above argument shows that $\bar{\Omega}(S)$ is a KB space.$^{11}$ By the construction, $\bar{\Omega}(S) \subset \tilde{\Omega}(S)$, and so

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$^{10}$For $i$, it is clear that $\tilde{\omega} \in P_i(\tilde{\omega})$ from the construction of $U^*(S)$.

$^{11}$Note that $\tilde{\Omega}(S)$ is also a continuous KB space.
\[ \tilde{\Omega}(S) = \tilde{\Omega}(S) \]. It means that \( \tilde{\Omega}(S) \) is closed.\footnote{Note that \( \tilde{\Omega}(S) \) is a continuous KB space.}

As a corollary of this result, we have the next theorem.

**Theorem 6.14.** *Except for KB spaces with redundant types, the space \( \Omega(S) \) is the universal continuous knowledge belief space.*

**Proof.** From the construction and Proposition 6.11, it is enough to show that \( \Omega(S) \) is a continuous knowledge belief space. By the definition, we know that \( \Omega(S) \subset \tilde{\Omega}(S) \). The above lemmas show that \( \tilde{\Omega}(S) \) is a continuous KB space. Therefore, \( \Omega(S) \supset \tilde{\Omega}(S) \), which means that \( \Omega(S) = \tilde{\Omega}(S) \). \( \square \)

### 6.3 The universal knowledge-belief space with a payoff irrelevant parameter space

Now we face the problem of redundant types referred to by Mertens-Zamir and given an example to by Ely-Peski [11]. One way to deal with such redundant types in Harsanyi type spaces is to introduce a *payoff irrelevant parameter space* \( C \) and construct the space of hierarchies of knowledge-belief over \( S \times C \) on the line of Liu [15]. He considered redundant types to be a result from a information missing in modeling. Some source of uncertainty, which does not affect the payoffs, are not incorporated in the model. As a result, the belief hierarchy over \( S \) is not enough to explain the type structure. To compensate the missing source of uncertainty, we have to add some payoff irrelevant parameter space to the basic uncertainty. Yokotani [23] showed that when Harsanyi type spaces are Polish, the two-valued parameter space \( C = \{0, 1\} \) is large enough to embed them into the universal type space over \( S \times C \). In this paper, we construct the universal type space with a payoff irrelevant...
parameter space in a analogous was of Yokotani.

First we introduce the next mathematical fact.

**Theorem 6.15.** *(Kechris [14], Theorem 4.14)* Every Polish space is homeomorphic to a subspace of the Hilbert cube $[0,1]^\mathbb{N}$.

Note that all spaces we are studying are compact metrizable. Since compact metrizable spaces are Polish, they are homeomorphically embedded to $[0,1]^\mathbb{N}$. This means that the Hilbert cube is rich enough to represent the information structure of any KB space. For this reason, we set $C \equiv [0,1]^\mathbb{N}$ as an exogenous parameter.

As in the same way, we can define the KB space and all concepts above over $S \times C$ too. Let $\mathcal{W} \equiv \langle W \subset S \times C \times \bar{T}, \ (\phi_i)_{i \in \mathbb{N}}, \ (\bar{P}_i)_{i \in \mathbb{N}} \rangle$ be a continuous KB space. Note that we can apply the same definition of S-homeomorphism to compare a KB space over $S$ and a one over $S \times C$ too.

The next proposition tells that, for any continuous KB space over $S$, we can always find its S-isomorphic continuous KB space over $S \times C$ which does not have redundant types.

**Proposition 6.16.** *For any continuous KB space $\mathcal{V}$ over $S$, there exists $\mathcal{W}$ over $S \times C$ that is S-homeomorphic to $\mathcal{V}$ and without redundant types.*

*Proof.* Let $\mathcal{V} \equiv \langle V \subset S \times T, \ (\lambda_i)_{i \in \mathbb{N}}, \ (P_i)_{i \in \mathbb{N}} \rangle$ be a continuous KB space over $S$. Since $V$ is compact metrizable, there exists a homeomorphism to a compact subspace of $C$. Let $\pi : V \to C$ be the homeomorphism. We construct $\mathcal{W}$ as follows. Let $\mathcal{W} \equiv \langle W \subset S \times C \times T, \ (\phi_i)_{i \in \mathbb{N}}, \ (\bar{P}_i)_{i \in \mathbb{N}} \rangle$ be such
Then, \( W \) is a continuous KB space over \( S \times C \) and S-homeomorphic to \( V \) because \( \pi \) is homeomorphism.

And since \( \pi \) is bijection, any \( t_i \neq t'_i \in \text{Proj}_T W \) knows different \( c \) and \( c' \in C \). It means that their first order hierarchies over \( S \times C \) are different, i.e. \( \gamma^1_i(t_i) \neq \gamma^1_i(t'_i) \). Thus there exists no redundant types in \( W \).

As a corollary of Theorem 6.14 and Proposition 6.16, we have the next theorem:

**Theorem 6.17.** Any countinuous KB space \( V \) over \( S \) is S-homeomorphic to a sub-continuous KB space of \( \Omega(S \times C) \).

**Theorem 6.18.** The continuous KB space \( \Omega(S \times C) \) is also a continuous KB space over \( S \).

*Proof.* We show that there exists a continuous KB space over \( S \) in which the set of Bayesian equilibria is the same as that of \( \Omega(S \times C) \) for every game \( g : S \times A \rightarrow X \). We construct a continuous
KB space $\hat{V} \equiv \langle \hat{V} \subset S \times T^*, (\lambda_i)_{i \in N}, (P_i)_{i \in N} \rangle$ as follows;

1. $\hat{V} = \text{Proj}_{(S \times T^*)} \Omega(S \times C)$,
2. $\forall i \in N, \forall t_i \in T^*_i, \lambda_i(t_i) = \text{Marg}_{(S \times T^*_i)} \phi_i(t_i)$,
3. $\forall i \in N, \forall t_i \in T^*_i, P_i(t_i) = \text{Proj}_{(S \times T^*_i)} \bar{P}_i(t_i)$.

Then, $\hat{V}$ is a continuous KB space over $S \times C$ and share the same equilibrium as $\Omega(S \times C)$ for every $g$. $\square$

Now we want to state our first main result. Before that, we have to slightly modify SCC because SCC is defined on the ex post type space, but we are now working on the epistemic type space. We define $\hat{F}$ is a social choice set (hereafter SCS) if $\hat{F} \subset F \equiv \{ f \mid f : \Omega(S \times C) \rightarrow X \}$.

Without loss of generality, we can identify a SCC $F$ to be a SCS $\hat{F}$ such that $\hat{F} \equiv \{ q \in F \mid \forall v \in \Omega(S \times C), q(v) \in F(s(v)) \}$. Based on this, we extend the definition of robust implementation to the SCC case. It is totally straightforward.

**Definition 6.19.** A mechanism $\mathcal{M}$ robustly implements a SCC $F$ if $\mathcal{M}$ Bayesian implements $\hat{F}$ in every Harsanyi type space $T$.

Then we have our first main result.

**Theorem 6.20.** A SCC $F$ is robustly implementable if and only if $\hat{F}$ is Bayesian implementable on $\Omega(S \times C)$.

**Proof.**

($\rightarrow$) Since $\Omega(S \times C)$ is a continuous KB space by Theorem 6.18, it is also a Harsanyi type space over $S$. Therefore the robust mechanism Bayesian implements $F$ on $\Omega(S \times C)$ by the definition of
robust implementation.

(←) From Theorem 6.17, any Harsanyi type space $\Lambda$ can be embedded to $\Omega(S \times C)$ as a KB space $\Lambda^{KB}$ whose knowledge set is the entire type space $\Lambda$. Now $\hat{F}$ is Bayesian implementable on $\Omega(S \times C)$, so there exists at least one Bayesian equilibrium. By Theorem 6.3, if $\hat{F}$ is Bayesian implementable on $\Omega(S \times C)$, it is Bayesian implemented on $\Lambda^{KB}$, i.e. $\Lambda$. Thus $F$ is robustly implementable.

□

7 Characterization of Implementation

Finally we give a necessary and sufficient condition for the Bayesian Implementation on $\Omega(S \times C)$. For notational convenience, we just use $\Omega$ for $\Omega(S \times C)$ in this section. Note that once we get all agents’ epistemic types $t$, we deduce what $s$ is. So the domain of SCS’s is represented as $F \equiv \{ f \mid f : \Pi_{i \in N} T_i \rightarrow X \}$, where $T_i$ is the space of the $i$’s epistemic types on $\Omega$. $^{13}$ For technical simplicity, we assume that $F \subset F$ is a countable set in this section.

First we have to define preliminary concepts for the characterization. A deception is just a way of lying.

Definition 7.1. A deception is a measurable mapping $\alpha_i : T_i \rightarrow T_i$. The family of the agents’ deception is denoted as $\alpha \equiv (\alpha_i)_{i \in N}$.

Definition 7.2. For each $z \in F$, $\tilde{t}_i \in T_i$, and $t \in T$, $z_{\tilde{t}_i}(t) \equiv z(\tilde{t}_i, t_{-i})$.

The agent $i$’s interim preference is given by Von-Neumann-Morgenstern preference.

$^{13}$To be mathematically precise, we have to eliminate the combinations of “null” states. However, as explained later, it does not bring significant changes.
Definition 7.3. For each $i \in N$, each $z$ and $z' \in \mathcal{F}$,

$$z R_i(t_i) z' \quad \text{if} \quad \int_{t_{-i} \in T_{-i}} zd\lambda^*_i(t_i) \geq \int_{t_{-i} \in T_{-i}} z'd\lambda^*_i(t_i).$$

Definition 7.4. A social choice set $F$ satisfies Bayesian incentive compatibility (hereafter, BIC) if, for all $q \in F$, $i \in N$, and $\tilde{t}_i \in T_i$,

$$\forall t_i \in T_i, \quad q R_i(t_i) q_{\tilde{t}_i}$$

The next one is a uncountable version of the closure concept used in Palfrey-Srivastava and Jackson.

Definition 7.5. (Closure) Let $\{\Psi_k\}_{k \in N}$ be a countable family of belief closed (sub-type) measurable spaces such that $\Omega = \bigcup_{k \in N} \Psi_k$. A social choice set $F$ satisfies closure (hereafter C) if, for all $\{q^k\}_{k \in N} \subset F$, there exists $z \in F$ such that, for each $k \in N$ and $t \in \Psi_k$, $z(t) = q^k(t)$.

We consider the following environment which is an interim version of bad outcome assumption by Bergemann-Morris.

Assumption 7.6. (Bad outcome assumption) For each $i \in N$, there exists $z^i* \in \mathcal{F}$ such that, for each $q \in F$, each $t_i \in T_i$, and each $\alpha$ there exists $r \in \mathcal{F}$ such that, (1) for each $t'_i \in T_i$, $q R_i(t'_i) r_{t_i}$, and (2) $(r \circ \alpha) P_i(t_i) (z^i* \circ \alpha)$.

We adopt Jackson’s monotonicity condition.

Definition 7.7. (Jackson [13]) A social choice set $F$ satisfies Bayesian monotonicity (hereafter
BM) if, for each \( q \in F \) and deception \( \alpha \),

\[
\text{if } \forall i \in N, \forall t_i \in T_i, \forall r \in F,
[\forall t_i' \in T_i, q R_i(t_i') r_{\alpha_i(t_i)}] \Rightarrow q \circ \alpha R_i(t_i) r \circ \alpha,
\]

then \( q \circ \alpha \in F \).

As we discuss later, the bad outcome assumption is enough to obtain a necessity and sufficient condition for SCFs. However, it is not enough for SCCs. We need another assumption.

**Assumption 7.8.** *(Economic environment)* For each \( q \in F \), and each \( t \in T \), there exists \( i \in N \) such that

\[
\exists r_i^j \in F \quad r^j P_i(t_i) q
\]

Under Assumption 7.6 and 7.8, we have the following theorem.

**Proposition 7.9.** Suppose \(|N| > 3\). If a social choice set \( F \) satisfies Bayesian incentive compatibility (BIC), closure (C), and Bayesian monotonicity (BM) on \( \Omega \), then \( F \) is Bayesian implementable on \( \Omega \setminus C^{null} \). Here \( C^{null} \) is a measurable set such that, for each \( i \) and \( t_i \in C_i^{null} \), \( \lambda_i^*(t_i)[C^{null}] = 0 \).

**Proof.** We construct a mechanism \( M \) as follows.

Let the agent \( i \)'s message space \( M_i \equiv T_i \times F \times \mathbb{Z}_+ \times \mathcal{F}^F \), and \( M \equiv \Pi_{i \in N} M_i \) where, for each \( q \in F \), \( m_i^q(q) \in \mathcal{F} \) such that, for each \( t_i' \in T_i \), \( q R_i(t_i') m_i^q(q) m_i^1 \). We make a partition on \( M \) as
follows;

\[D_0 \equiv \{ m \in M \mid \forall i \in N, \exists q \in F, m_i = (., q, 1, .) \},\]

\[D_1^i \equiv \{ m \in M \mid \exists i \in N, \forall j \neq i, \exists q \in F, m_j = (., q, 1, .), \text{ and } m_i^2 \neq q \text{ or } m_i^3 > 1 \},\]

\[D_1 \equiv \bigcup_{i \in N} D_1^i,\]

\[D_2 \equiv \{ m \in M \mid m \notin D_0 \cup D_1 \}\]

The payoff function \( g : M \rightarrow A \) is given as;

\[
g(m) = q(m^1) \quad \text{if } m \in D_0,
\]

\[
g(m) = m_i^2(m^1)(1 - \frac{1}{1 + m_i^2}) + z^i s(\frac{1}{1 + m_i^2}) \quad \text{if } m \in D_1 \text{ and } \forall t_i^t \in T_i, q R_i(t_i^t) m_i^2 m_i^1,
\]

\[
g(m) = m_i^4(q)(m^1)(1 - \frac{1}{1 + m_i^2}) + z^i s(\frac{1}{1 + m_i^2}) \quad \text{if } m \in D_1 \text{ and } \exists t_i^t \in T_i, m_i^2 m_i^1, P_i(t_i^t) q,
\]

\[
g(m) = m_k^2(m^1)(1 - \frac{1}{1 + m_k^2}) + z^k s(\frac{1}{1 + m_k^2}) \quad \text{if } m \in D_2,
\]

where \( k \) is the agent such that \( \forall j \in N, m_k^3 \geq m_j^3 \).

We show that this mechanism implements \( F \) by showing the following two lemmas.

**Lemma 7.10.** If \( F \) satisfies (BIC), then, for each \( q \in F \), there exists a Bayesian equilibrium \( \sigma \) of \( (\Omega, M, g) \) such that \( q = g(\sigma) \).

**Proof.** Fix \( q \in F \). Suppose that \( F \) satisfies (BIC). Then, for each \( i \in N, t_i \in T_i, \) and \( t_i^* \in T_i \), we have that \( q R_i(t_i) q_i^* \).

Now, for each \( i \in N, \) let \( \sigma_i \) be the agent \( i \)'s strategy such that, for each \( t_i \in T_i, \sigma_i(t_i) = (t_i, q, 1, .) \).

Then \( g(\sigma) = q \). We show that \( \sigma \) is a Bayesian equilibrium.

33
Lemma 7.11. The following lemma completes the proof.

Let \( \tilde{\sigma}(t_i) = (\alpha_i(t_i), \, q, \, 1, \, . \,) \), where \( \alpha_i \) is a deception. Then, \( g(\tilde{\sigma}, \sigma_{-i}) = q_{\alpha_i} \). Since, for each \( i \in N, \, t_i \in T_i \), and \( t^*_i \in T_i \), \( q \, R_i(t_i) \, q_{t^*_i} \), it holds that \( q \, R_i(t_i) \, q_{\alpha_i(t_i)} \). It means that \( \sigma_i \) is better than \( \tilde{\sigma}_i \) given \( \sigma_{-i} \).

(Case 1) Let \( \tilde{\sigma}_i(t_i) = (\alpha_i(t_i), \, r, \, 1, \, . \,) \) and \( r \neq q \). If there exists \( t'_i \in T_i \) such that \( r_{\alpha_i(t_i)}(t'_i) \, q \), then, by the construction of \( g \), \( g(\tilde{\sigma}, \sigma_{-i}) = q_{\alpha_i} \). Therefore, the above argument implies that \( \sigma_i \) is better than \( \tilde{\sigma}_i \) given \( \sigma_{-i} \). Next we assume that there exists \( \hat{t}_i \in T_i \) such that, for each \( t'_i \in T_i \), \( q \, R_i(t'_i) \, r_{\alpha(i_i)} \). Then, for each \( t_{-i} \in T_{-i}, \, g(\hat{\sigma}(\hat{t}_i), \sigma_{-i}(t_{-i})) = r(\alpha_i(\hat{t}_i), t_{-i})(1 - \frac{1}{1+n_i}) + z^i*(\frac{1}{1+n_i}). \)

So \( g(\hat{\sigma}, \sigma_{-i}) \) and \( r(\alpha_i(\hat{t}_i), t_{-i})(1 - \frac{1}{1+n_i}) + z^i*(\frac{1}{1+n_i}) \) are equivalent in expected preference at \( \hat{t}_i \). On the other hand, \( q \, R_i(\hat{t}_i) \, r_{\alpha(i_i)} \) by the assumption. Therefore

\[
q \, R_i(\hat{t}_i) \, r_{\alpha(i_i)} = r_{\alpha(i_i)} \, R_i(\hat{t}_i) \, r(\alpha_i(\hat{t}_i), t_{-i})(1 - \frac{1}{1+n_i}) + z^i*(\frac{1}{1+n_i}).
\]

It means that \( q \, R_i(\hat{t}_i) \, g(\tilde{\sigma}, \sigma_{-i}) \).

The above arguments means that, for each \( i \) and \( t_i \), there is no incentive to deviate from \( \sigma_i \). Thus \( \sigma \) is a Bayesian equilibrium. \( \Box \)

The following lemma completes the proof.

**Lemma 7.11.** If \( F \) satisfies (C) and (BM), then, for each Bayesian equilibrium \( \sigma \), there exists \( z \in F \) such that, for each \( t \in \Omega \setminus C^{null} \), \( z = g(\sigma) \).

**Proof.** Let \( \sigma \) be a Bayesian equilibrium for the game \((\Omega, \, g, \, M)\). Let \( \alpha_i : T_i \rightarrow M^1_i \) be the deception by the agent \( i \) such that, for each \( t_i \), \( \alpha_i(t_i) = \sigma^1_i(t_i) \).

Let \( B_r^1 \equiv \{t_i \in T_i \mid \sigma_i(t_i) = (\alpha_i(t_i), \, r, \, 1, \, . \,) \} \) and \( B_r \equiv \{t \in T \mid \forall i \in N, \, \sigma_i(t_i) = (\alpha_i(t_i), \, r, \, 1) \} \).
(Case 1) Suppose that, given the other agents’ strategy profiles $\sigma_{-i}$, for some $t_i \in T_i$, $\sigma^3_i(t_i) > 1$ is the best strategy. Then, for each $t_{-i} \in T_{-i}$, the resulting message $\sigma(t_i, t_{-i}) \in D_1 \cup D_2$. Therefore, under Assumption 7.6, the agent $i$ gets better by increasing $m^3_i$ to infinity. It means that $\sigma^3_i(t_i) > 1$ cannot be the best response. Thus at equilibrium, $\sigma^3_i(m_i) = 1$.

(Case 2) Suppose that, at an equilibrium strategy profile $\sigma$, for some $t_i \in T_i$, $\lambda^*(t_i)[\{t_{-i} | \sigma^2_{-i}(t_{-i}) \neq \sigma^2_i(t_i)\}] > 0$. By the above argument, $\lambda^*(t_i)[\{t_{-i} | \sigma^3_{-i}(t_{-i}) = 1\}] = 1$. In this case, if $\sigma^3_i = 1$, then $m^3_i \to \infty$ is better if $\sigma(t) \notin D_0$. And if $\sigma(t) \in D_0$, $i$ gets $\sigma^2_i(t_i)$. Therefore, by the message $\tilde{m}_i(t_i) = (\sigma^1_i(t_i), \sigma^2_i(t_i), \infty, \tilde{m}_i^4)$ such that $\tilde{m}_i^4(\sigma^2_i(t_i)) = \sigma^2_i(t_i)$, he gets better. It cannot be an equilibrium.

As a result, in the equilibrium, it must be satisfied that

For each $i \in N$, and $t_i \in T_i$,

$$\exists q \in F, \quad \sigma_i(t_i) = (\alpha_i(t_i), q, 1, . ), \text{ and }$$

$$\lambda^*(t_i)[\{t_{-i} | \sigma_{-i}(t_{-i}) = (\alpha_{-i}(t_{-i}), q, 1, . )\}] = 1.$$ 

Therefore $\Omega = \bigcup_{r \in F} B_r \cup C^{null}$. Since $F$ is countable, $\bigcup_{r \in F} B_r$ is a countable union of measurable sets, which means both $\bigcup_{r \in F} B_r$ and its complement $t \in C^{null}$ are measurable. By Closure, there exists $\tilde{q} \in F$ such that, for $t \in \Omega \setminus C^{null}$, (1) $\tilde{q}(t) = r(t)$ if $t \in B_r$ and (2) $g(\sigma) = \tilde{q} \circ \alpha$.

We want to show that $\tilde{q} \circ \alpha \in F$. To do this, we use contradiction. Suppose that there is no
$z \in F$ such that $z = \tilde{q} \circ \alpha$. Then we apply Bayesian monotonicity to obtain that

$$\exists i \in N, \exists \tilde{t}_i \in T_i, \exists r \in \mathcal{F},$$

$$\forall t'_i \in T_i, \ \tilde{q} \circ R_i(t'_i) \circ r_{\alpha_i(t_i)} \text{ and } r \circ \alpha P_i(t_i) \circ \tilde{q} \circ \alpha.$$  

Therefore,

$$\tilde{\sigma}_i \equiv \begin{cases} 
\sigma(t_i) & \text{for } \forall t_i \neq \tilde{t}_i \\
(\alpha_i(\tilde{t}_i), r, v_i) & \text{for } \tilde{t}_i 
\end{cases}$$

where $v_i$ is a sufficiently large number.

Then, this new strategy $\tilde{\sigma}_i$ is better than $\sigma_i$ for the agent $i$ given $\sigma_{-i}$. It contradicts that $\sigma$ is a Bayesian equilibrium.

$\square$

Next we show that the inverse direction of the theorem.

**Proposition 7.12.** If $F$ is Bayesian implementable on $\Omega$, it satisfies (C), (BIC), and (BM).

**Proof.** Suppose that a mechanism $\mathcal{M} \equiv (M, g)$ implements a SCF $F$ on $\Omega$. Let

$$\tilde{F} \equiv \{ z \in \mathcal{F} | \exists \sigma \text{ s.t. } \sigma \text{ is a Bayesian equilibrium on } \mathcal{M}, \text{ and } z = g(\sigma) \}.$$  

Since $F$ is implemented by $\mathcal{M}$, it must hold that $F = \tilde{F}$.

Since $\tilde{F}$ is the set of Bayesian equilibrium, the condition (C) is satisfied. We show that the condition (IC) is satisfied below. Let $i \in N, q \in F$ and $\sigma$ be a Bayesian equilibrium such that $g(\sigma) = q$. Now let the agent $i$’s strategy $\tilde{\sigma}_i$ be such that there exists $t^*_i \in T_i$ and, for each $t_i \in T_i$, $\tilde{\sigma}_i(t_i) = \sigma_i(t^*_i)$.
Since $\sigma_i$ is the equilibrium strategy,

$$\forall t_i \in T_i, \ g(\sigma) \ R_i(t_i) \ g(\hat{\sigma}, \sigma_{-i}).$$

We have $g(\sigma) = q$ and $g(\hat{\sigma}, \sigma_{-i}) = q_{t_i^*}$. Thus

$$\forall t_i \in T_i, \ q \ R_i(t_i) \ q_{t_i^*}.$$ 

The condition (BIC) is satisfied.

Next we show (BM). Let $q \in F$ and $\sigma$ be as before. Suppose that, for some deception $\alpha$, there is no $z \in F$ such that $z = q \circ \alpha$. Since $\tilde{F} = F$, $q \circ \alpha \notin \tilde{F}$. Therefore, there exist $t^* \in T$ where, for some agent $i$, $\sigma_i(t_i^*)$ is not best response. It means that there exists a better response $\tilde{m}_i \in M_i$ for the agent $i$ at $t_i^*$ given $\sigma_{-i}$. Let $\tilde{\sigma}_i$ be such that, for each $t_i \in T_i$, $\tilde{\sigma}_i(t_i) = \tilde{m}_i$. And let $r \equiv g(\tilde{\sigma}_i, \sigma_{-i})$. Then we have that $g[(\tilde{\sigma}_i, \sigma_{-i}) \circ \alpha] \ P_i(t_i^*) \ g(\sigma \circ \alpha)$. It means that $r \circ \alpha \ P_i(t_i^*) \ q \circ \alpha$. On the other hand, for each $t_i \in T_i$, $g(\sigma) \ R_i(t_i) \ g(\tilde{m}_i, \sigma_{-i})$ because $\sigma$ is a Bayesian equilibrium. Therefore, for each $t_i \in T_i$, $q \ R_i(t_i) \ r$. Note that, for each $t_i \in T_i$, $\tilde{\sigma}_i(t_i) = \tilde{m}_i$. Therefore, $r_{\alpha_i} = r$. From the above equations, we have that, for each $t_i \in T_i$, $q \ R_i(t_i) \ r_{\alpha_i(t_i)}$. Thus (BM) is satisfied. □

References


[15] Liu, Q. (2009),


