Coalitional stochastic stability

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Abstract

This paper takes the idea of coalitional behaviour - groups of people occasionally acting together to their mutual benefit - and incorporates it into the framework of evolutionary game theory that underpins the social learning literature. An equilibrium selection criterion is defined which we call coalitional stochastic stability (CSS). This differs from existing work on stochastic stability in that profitable coalitional deviations are given greater importance than unprofitable single player deviations. CSS states are characterized for all normal form games and this characterization is shown to give a lexicographic ranking of efficiency and risk-dominance for 2x2 games, although a further example demonstrates that efficient selection is not guaranteed in larger games despite the Pareto improving nature of the deviations underlying CSS. Successive examples illustrate how the model can be used to give theories of: (i) The persistence and direction of inequitarian social norms; (ii) How collusive behaviour between firms can promote (as well as inhibit) price competition in Bertrand models; (iii) How one of the problems affecting stochastic stability methods - large expected time to convergence to stable states - can be mitigated using a coalitional approach, explaining how social change can occur in reasonable timescales and new technologies and standards can achieve rapid dominance, spending little time at heterogeneous steady states.

Keywords: Stochastic stability, learning, coalition, lexicographic, contract, experimentation, time to convergence.

JEL classifications: C71, C72, C73

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1 Introduction

In their seminal papers in evolutionary game theory Foster and Young (1990) and Kandori et al. (1993) introduce the idea of stochastic stability: a method of equilibrium selection which assesses the robustness of equilibria by measuring their resilience to random errors in players’ actions. The model of Foster and Young features a continuous dynamic whereas Kandori et al. look at discrete dynamics and give a graphical approach to finding the stochastically stable states. Young (1993) applies these ideas to the evolution of social norms in a discrete time dynamic process which he calls adaptive play. Under adaptive play, players repeatedly play an \( n \) player game \( \Gamma \). We note that Young interprets adaptive play as modelling situations where each player is actually a representative agent picked at random from some population of similar agents. This interpretation is also a valid possibility for this paper, but for the sake of clarity we stick to using the term ‘player’ to describe a position in a game, whether it is the same agent repeatedly playing or various agents plucked randomly with replacement from some underlying population.

Players follow a process whereby they play best responses to a distribution over the actions played by the other players, where the distribution is determined by sampling \( s \) actions from the previous \( m \) actions played by the other players. This defines a Markov process where the states of the process are defined by the actions taken by each of the players in the previous \( m \) periods. If \( s \) is small enough relative to \( m \) and the game is weakly acyclic the game converges almost surely to a convention where each player has played a strategy from a pure strategy Nash equilibrium of \( \Gamma \) for as long as anybody can remember. Young introduces random shocks to the system which can be interpreted as random mistakes made by players in implementing their strategies. As long as there is a positive probability of every combination of strategies being played by mistake the perturbed system then defines an aperiodic and irreducible Markov process. Young shows that as the probability of a random shock \( \epsilon \) approaches zero, the system spends almost all of its time in a subset of the conventions of the games. He calls such conventions stochastically stable.
Here it is argued that stochastic stability can lead to unrealistic results in games where coalitional behaviour can be expected by players. Furthermore, we define coalitional stochastic stability and claim it can lead to a more realistic equilibrium selection in certain settings, particularly in games with more than 2 players. Coalitional stochastic stability uses the possibility of deviations by groups of players as an equilibrium selection device in evolutionary models of social learning. This paper describes, justifies and illustrates the uses of this innovation, giving simple and intuitive methods of finding CSS states. Applications of the theory are given throughout the paper, illustrating how CSS can be used to give theories of: (i) The persistence and direction of inegalitarian social norms; (ii) How collusive behaviour between firms can promote (as well as inhibit) price competition in Bertrand models; (iii) How one of the problems affecting stochastic stability methods - large expected time to convergence to stable states - can be substantially mitigated using a coalitional approach, explaining how social change can occur in reasonable timescales and new technologies and standards can achieve rapid dominance, spending little time at heterogeneous steady states.

Section 2 of this paper summarizes the two areas of related literature and gives the motivation and contribution of this paper in more detail. Section 3 describes the basic model of adaptive play on which this paper builds. Section 4 gives a motivating example demonstrating how random error based stochastic stability can prove an inadequate tool and how coalitional considerations can be important. Section 5 describes the concept of Coalitional Stochastic Stability. Section 6 includes the fundamental characterization results of the paper and several examples and applications. Section 7 concludes. Formal proofs are given in the Appendix.

2 Related literature

There are two strands of literature which this paper bridges. Below I give a brief summary of both of them, followed by a description of the motivation and contribution of this paper.
2.1 Coalitional behaviour

There exists a large literature in cooperative game theory on the behaviour of coalitions. For a survey the reader is referred to Peleg and Sudholter (2003). Aumann (1959) gives the concept of a ‘strong equilibrium’ - an equilibrium where no subset of players would want to agree to change their profile of strategies to another profile, holding the strategies of all players not in that subset fixed. This equilibrium concept can be argued to correspond to situations where coordination between any subset of players is possible without players outside the subset being aware of it. As the name suggests this is a very strong equilibrium notion and often will not exist. The concept can be weakened to that of k-strong equilibrium where only coalitions of size k or lower have to have their incentive constraints satisfied but still existence is not guaranteed.\footnote{Nash equilibrium is a special case of k-strong equilibrium where k=1.} Bernheim et al. (1987) attempt to address the issue of robustness to coalitional deviations through their concept of coalition proof equilibrium, the idea of which is that equilibria need to be robust to a set of players deviating only if that set of players is itself robust to any further deviations by subsets of its constituent players. Bernheim et al. argue that this equilibrium concept can be understood intuitively to lead to outcomes which could be reached if all players seated in a room reached an agreement, following which the players leave the room one by one, and no matter in what order they leave the room there will never be a subset of players remaining in the room who would agree to play differently to what was agreed with all players present. Konishi and Ray (2003) look at the issue of coalition formation in a dynamic setting with farsighted agents, showing that if characteristic functions are incorporated into the rules governing the dynamic process then they can choose such a process which always selects payoffs in the core of the underlying game if the core is a singleton. Ambrus (2006) defines and analyzes a concept of coalitional rationalizability: the idea that subsets of players will refrain from playing certain strategies if it is in their interests to do so. Luo and Yang (2009) extend this concept to situations where players use Bayesian updating to calculate expected payoffs.
2.2 Stochastic stability

Whereas previous concepts of equilibrium stability such as asymptotic stability or evolutionary stable strategies (Smith and Price (1973)) focus on robustness to single errors (mutations) in strategies, Young (1993) and Kandori et al. (1993) introduce the possibility of multiple ‘random errors’ in strategies chosen and show that although there may be several stationary states in a dynamic process, some of them may be more robust to such errors than others, and that if the probability of a random error in the very long run becomes very small, then the state which is most robust to such errors will be observed almost all of the time. Kandori et al. (1993) and Young (1993) predict that in 2x2 games when there are two strict Nash equilibria then the risk dominant equilibrium will be selected. It has been noted by Ellison (1993) that the time taken for random errors to cause a switch between stationary states of the underlying process can be very long, although Ellison (2000) notes that if movement between states of the underlying process primarily takes place between states which are ‘close’ to one another then the time required for switches can be lower. Bergin and Lipman (1996) prove a kind of folk theorem for stochastic stability, that is they show that any stable state of the unperturbed dynamic process can be selected with appropriately chosen state-dependent mutation rates. van Damme and Weibull (2002) recover some of the predictive power of the theory by assuming that avoiding mistakes is costly to players. They endogenize the random error probabilities so that players will pay more to avoid making mistakes which are more costly to them and give conditions under which the probabilities of random errors by any player at any state are of the same order of magnitude and remain so as limits are taken. They show that under these conditions the results of Young are recovered. Young (1998a) shows that under his uniform error stochastic stability process there is a preference for fairness in contracts agreed to between two players and that the contract selected corresponds to the Kalai-Smorodinsky bargaining solution. Naidu et al. (2010) analyse a model of contracting where movement from one Pareto efficient contract to another is only caused by errors on the part of the player who stands to gain
from the move. They justify this by invoking a level of foresight on the part of the agents, who know that there is a better contract available and thus ‘intentionally’ make mistakes so as to lead to a better conventional contract for themselves.

2.3 Motivation and contribution of this paper

2.3.1 Individual rationality

Instead of perturbing rationality to obtain predictions in games, we focus on perturbing individuality. We look at environments where from time to time players may meet and agree to jointly coordinate their actions. We are particularly interested in what happens as such coalitional activity becomes rare. To this end we develop a stochastic stability notion which we call coalitional stochastic stability (CSS) which is based on resilience to coalitional behaviour, unlike traditional stochastic stability notions which are based on resilience to random errors.

2.3.2 Why model coalitional behaviour as rare?

It is one thing to analyse the behaviour of a dynamic where coalitional behaviour is possible. It is another thing to analyse the predictions of a model where coalitional behaviour becomes infinitesimally likely in the limit. Aside from pure academic interest here are two reasons why we should be interested. Firstly, people deal with a lot of games in their everyday lives and the amount of time they devote to any given one is by necessity limited. It is not unreasonable to think that it is quite a rare event that two or more people get together and discuss any particular aspect of their lives and credibly agree to make the necessary changes in strategy to better their outcomes. It is also not unreasonable to think that the more people required to agree to changes in behaviour, the harder these changes are to accomplish. Secondly, for the results of this paper it is not necessary that individual strategic switching occur with positive probability in the limit, merely that it is much more likely than coalitional strategic switching. Thus, any type of strategic change can
be viewed as a rare event, a fair assumption given the stasis observed in people’s behaviour in most aspects of their lives.

2.3.3 CSS justifies ‘experimentation’

As described above, several models have attempted to model random deviations as ‘experimentation’ by players. The problem here is that when in an equilibrium state such experimentation by a single player is always going to damage the payoff of the experimenting player in the short term. To deal with this, justifications for experimentation have to endow the players with some amount of foresight, for example by suggesting that players will be more likely to experiment when at states which give them relatively poor payoffs because they feel that there must be ‘something better out there’. Such justifications sit uneasily with the myopic nature of the rest of the adaptive learning process. CSS however, easily justifies interpretations of deviations as experimentation, as experimenting players can achieve higher payoffs by participating in a coalitional deviation. CSS can thus incorporate the idea of experimentation into stochastic stability notions without departing from the myopia of standard adaptive processes.

2.3.4 CSS can significantly lower convergence times

One of the problems with stochastic stability \(^2\) is the very long times it can take to move between stationary states of the unperturbed process. The reason for this is that players typically need to make several mistakes to push the process from one state to another, the probability of each random error occurring is very small, and the probability of several random errors occurring is much smaller still. Moreover, it is frequently the case that several mistakes need to be made by a single player (or class of players) to push the system to another state. This requires a player to make an error which damages his payoff then continue to make that error several times. With CSS however, we have every reason to believe that a profitable joint deviating strategy will be continued in future periods. If a coalition of players have

\(^2\)See for example Ellison (1993)
tried something and it works, then why not try it again? This paper will show how persistence of profitable coalitional deviations can lead to a considerable reduction in time taken to move between different states of the dynamic process. Unsurprisingly, it cannot be stated unambiguously that coalitional behaviour will always lower convergence times: it is also possible that the ability of players to jointly alter their strategies can have a conservative effect and assist in maintaining the current stationary state of the unperturbed process. For 2-dimensional local interaction on a lattice as in Ellison (2000) we demonstrate that both effects are possible and that the predominant effect is determined by the parameters of the model.

2.3.5 Realistic modeling of social change

The combination of myopic behaviour with the assumption that coalitional behaviour is more likely to take place with smaller coalitions of players than with larger coalitions can be used to explain aspects of social change. It gives an explanation, for instance, of why revolutionary social movements (large coalitional deviations) will typically have a short life span before breaking down (small coalitional deviations) into something other than what was originally intended.

3 Basic model

This paper shall follow closely the methods and notation of Young (1993).\textsuperscript{3} Take an n-player game $\Gamma$ with finite strategy sets $X_1, \ldots, X_n$; $X = \prod X_i$; and payoffs given by $\pi_i : X \to \mathbb{R}$. The action taken by player $i$ at time $t$ is denoted $x_i^t$. The action profile played at time $t$ is denoted $x^t = (x_1^t, \ldots, x_n^t)$. The state of the system is given by the actions played in the last $m$ periods and is denoted $h_t = (x^{t-m+1}, \ldots, x^t)$. The higher the value of $m$ the longer the memory of the players. $X^m$ denotes the set of all possible states. The system is taken to start at an arbitrary state: it is assumed at least $t$ periods have already elapsed since the beginning of time.

\textsuperscript{3}See also Young (1998a), Young (1998b).
We define a Markov process $P^0$ on $X^m$ as follows:

- From state $h_t$ player $i$ draws a random sample of $s$ actions out of the previous $m$ actions taken by each of the other players. These $n-1$ samples are drawn independently. The sample distribution of $j$’s actions in $i$’s sample is denoted $\hat{p}_{t ij}$ and we write $\prod_{j \neq i} \hat{p}_{t ij} = \hat{p}_{t-i}$.

- Player $i$ plays a best response to $\hat{p}_{t-i}$. We denote the set of such best responses by $B_i(\hat{p}_{t-i})$. If there exist tied best responses they are played with equal probability. The actions all players take define $x_{t+1}$ and thus the next state $h_{t+1}$.

A convention is defined as a state $h_t = (x_{t-m+1}, \ldots, x^t)$ where $x_{t-m+1} = x_{t-m+2} = \ldots = x^t = x^*$ and $x^*$ is a strict Nash equilibrium of the underlying game $\Gamma$. It is clear that under the process $P^0$ once a convention is reached it will be sustained forever. However, Young defines a perturbed version of the process $P^\epsilon$ where with probability $1-\epsilon$ a player plays a best response as per $P^0$. With probability $\epsilon$ he instead makes an ‘error’ and plays a random action from a distribution with full support over his possible actions. In the limit as $\epsilon \to 0$ only some conventions are played with positive probability in the long run — they are stochastically stable.

4 Motivating example

In his paper dealing with conventional contracts Young (1998a) gives the example of The Marriage Game. In this game a woman and a man have the option of taking control (TC), sharing control (SC) or ceding control (CC). Those who feel the concept of coalitional deviations in social learning to be self-motivating can skip straight to section 5.

5Note that this is not a particularly esoteric example and can be used to model a number of situations where two agents have to agree on a contract. Instead of a man and a woman in the example we could have a real estate company and a building contractor, with payoffs representing the share of each party’s surplus generated by the agreed contract.
Young (1998b) shows that the stochastically stable outcome of this game when the sample size is sufficiently large is for men and women to share control. The reason for this is that “conventions with extreme payoff implications are relatively easy to dislodge” because one of the players is “dissatisfied compared to what they could get under some other arrangement”. It does not take many stochastic shocks to create an environment in which they prefer to try something different.

Here I expand the marriage game to include more than 2 players. Specifically I look at a version of the game where there is one man and $n$ women playing the game. The man receives his coordination payoff as long as at least one woman coordinates with him. If more than one woman correctly coordinates with the man, each woman to do so “marries” (i.e. receives her coordination payoffs) with equal probability. Players have the same actions available as in the game above. Payoffs are specifically:

$$
\pi_M(TC) = \begin{cases} 
5, & \text{if at least one woman plays CC} \\
0, & \text{otherwise}
\end{cases}
$$

$$
\pi_M(SC) = \begin{cases} 
3, & \text{if at least one woman plays SC} \\
0, & \text{otherwise}
\end{cases}
$$

$$
\pi_M(CC) = \begin{cases} 
1, & \text{if at least one woman plays TC} \\
0, & \text{otherwise}
\end{cases}
$$

In the following expressions $d$ is the number of women who play the same action as the woman whose payoff is in question (inclusive).

$$
\pi_W(TC) = \begin{cases} 
5/d, & \text{if the man plays CC} \\
0, & \text{otherwise}
\end{cases}
$$
\( \pi_W(\text{SC}) = \begin{cases} 
\frac{3}{d}, & \text{if the man plays SC} \\
0, & \text{otherwise} 
\end{cases} \)

\( \pi_W(\text{CC}) = \begin{cases} 
\frac{1}{d}, & \text{if the man plays TC} \\
0, & \text{otherwise} 
\end{cases} \)

In Young (1998b) stochastically stable states are found by computing the resistances between recurrent classes\(^6\) of \( P^0 \). These are defined as the smallest number of stochastic shocks required by the process \( P^e \) in order to move from one recurrent class of \( P^0 \) to another. Figure 1 shows what Young calls the reduced resistances of the 2-player marriage game. These are the resistances divided by the sample size \( s \).

Young also defines a concept called stochastic potential. If each recurrent class of the process is drawn as a vertex on a graph, then for a given recurrent class \( i \) we can draw a spanning tree such that from every other recurrent class \( j \neq i \) there is a unique path from \( j \) to \( i \). The resistances of the edges of this graph can be summed. The stochastic potential of recurrent class \( i \) is then defined as the minimum of these sums across all possible spanning trees. The stochastically stable recurrent classes are then shown in Theorem 2 of Young (1993) to be the classes with the lowest stochastic potential.

We wish to calculate the resistances in the n-woman marriage game. There are 3 absorbing states in the n-woman marriage game. I shall call these \( \sigma, \, \varphi, \, \varphi^c \) representing male, shared and female control respectively (e.g. In \( \sigma \) the man plays TC and all the women play CC). There are basically two types of possible transition here: those with M’s payoff increasing and those with M’s payoff decreasing. Here we examine one instance of each (from which follows the general result).

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\(^6\)A recurrent class of a Markov process is a set of states of the process \( \Xi \) such that for any \( \omega_1, \omega_2 \in \Xi \) there is positive probability when starting in state \( \omega_1 \) of reaching state \( \omega_2 \) in a finite number of periods and for any \( \omega_3 \notin \Xi \) there is zero probability when starting in state \( \omega_1 \) of reaching state \( \omega_3 \).
Figure 1: Reduced resistances for 2 player marriage game

4.1 M’s payoff increasing

Transition: $\varphi \rightarrow \sigma$

Firstly we ask how many times women must mutate for this transition to take place. Say one woman out of the $n$ women undergoes mutations. We call $r$ the number of mutations necessary for the transition to take place with no further mutations. This is the number of mutations required for M to think it worthwhile to play TC after sampling the actions of the women. We examine M’s expected payoffs when he samples $r$ plays of CC from the woman in question (the other women do not mutate and continue to play TC).\(^7\)

\[
E[\pi_M(CC)] = 1
\]

\[
E[\pi_M(TC)] = 5\frac{r}{s}
\]

\(^7\)If only one woman mutates then each additional mutation will add $\frac{1}{s}$ to the man’s subjective probability that TC will be played by at least one woman. Any mutations by a different woman will only increase this probability by $\frac{1}{s}$ multiplied by the probability that the first woman does not play TC.
Hence M might switch from CC to TC when

\[ r \geq \frac{1}{5} \]

In this way we find candidates for reduced resistances \( r_r = \frac{r}{s} \):

<table>
<thead>
<tr>
<th>Transition</th>
<th>( r_r ? )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varnothing \rightarrow \sigma )</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>( \varnothing \rightarrow \varnothing )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>( \varnothing \rightarrow \sigma )</td>
<td>( \frac{3}{5} )</td>
</tr>
</tbody>
</table>

But now we have to ask if such a transition could instead be caused by rational behaviour by the women. This is never the case in the two player marriage game for transitions that improve M’s payoff, but can be the case for the n-woman game as there is a first mover advantage in that if a woman guesses correctly the action of the man and the other women fail to do so a higher relative payoff ensues as it is not shared with the other women.

So say the man has suffered \( r \) mutations from CC to TC. If the women sample all of these mutations then:

\[
E[\pi_W(TC)] = \frac{5(s - r)}{n s}
\]

\[
E[\pi_W(CC)] = \frac{1r}{s}
\]

Hence W might switch from TC to CC when

\[ r \geq s \frac{5}{n + 5} \]

The transition of M to TC can follow without any further mutations.

In this way we find further candidates for reduced resistances:

<table>
<thead>
<tr>
<th>Transition</th>
<th>( r_r ? )</th>
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<tbody>
<tr>
<td>( \varnothing \rightarrow \sigma )</td>
<td>( \frac{5}{n+5} )</td>
</tr>
<tr>
<td>( \varnothing \rightarrow \varnothing )</td>
<td>( \frac{5}{3n+5} )</td>
</tr>
<tr>
<td>( \varnothing \rightarrow \sigma )</td>
<td>( \frac{3}{n+3} )</td>
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</tbody>
</table>
We conclude that reduced resistances for the following transitions are:

<table>
<thead>
<tr>
<th>Transition</th>
<th>$r_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi \rightarrow \sigma$</td>
<td>$\min(\frac{1}{5}, \frac{5}{n+5})$</td>
</tr>
<tr>
<td>$\varphi \rightarrow \varphi$</td>
<td>$\min(\frac{1}{3}, \frac{5}{3n+5})$</td>
</tr>
<tr>
<td>$\sigma \rightarrow \sigma$</td>
<td>$\min(\frac{2}{5}, \frac{3}{n+3})$</td>
</tr>
</tbody>
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4.2 M’s payoff decreasing

Here we need only examine transitions of the type where M mutates and W react rationally to this mutation. Following an identical procedure to the second part of the subsection immediately preceding we obtain the following reduced resistances:

<table>
<thead>
<tr>
<th>Transition</th>
<th>$r_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \rightarrow \varphi$</td>
<td>$\frac{1}{5n+1}$</td>
</tr>
<tr>
<td>$\sigma \rightarrow \sigma$</td>
<td>$\frac{1}{3n+1}$</td>
</tr>
<tr>
<td>$\varphi \rightarrow \varphi$</td>
<td>$\frac{3}{5n+3}$</td>
</tr>
</tbody>
</table>

So we have obtained resistances for every possible transition in our n-woman marriage game.

It is apparent from the diagram and the argument above that there are two effects which make the n-woman marriage game different to the 2-player marriage game. The first effect is that even when one woman has mutated away from the current convention, there are still sometimes women who remain playing that convention. This increases the expected payoffs to M from sticking with the existing convention and makes the resistances of transitions which are payoff improving for M higher than in the 2-player game. The second effect is that lower payoffs for the women in the n-woman marriage game lead to a greater willingness of individual women to experiment with different actions when they observe behaviour by M that is not in keeping with the current convention. For big enough $n$ this effect will come to dominate the first effect for transitions which improve M’s payoff. However, the second effect is greater for transitions which decrease M’s payoff than for transitions
Figure 2: Reduced resistances for n-woman marriage game

which increase his payoff. The result is that, as \( n \) increases, stochastic stability selects conventions which give \( M \) lower payoffs. In fact for \( n \geq 2 \), \( \Phi \) is the stochastically stable state. This is counterintuitive: it would seem to be a reasonable real world assumption that increased numbers of women in the marriage market would lead to greater market power for \( M \) and allow him to extract higher payoffs.

4.3 Why this peculiar outcome occurs

Results like this continue to occur even when the number of contracts available to the man and the women increases markedly (eg. to sharing 100 units in integer increments). These strange results are driven by the probabilities given to mutation patterns in each period. Within the framework of Young (1993) payoff improving deviations by a single player have a high probability of being realized. Payoff decreasing deviations by a single player have a probability of order \( \epsilon \) of being realized. Deviations by two players such that either deviation on its own would be detrimental to the deviator’s payoff have a probability of order \( \epsilon^2 \) of being realized, whether or not the deviations taken together offer a pareto improvement to the deviators. Thus as \( \epsilon \rightarrow 0 \), joint deviations by two players which offer payoff improvements
become infinitely unlikely compared to unprofitable deviations by a single player. This is clearly not satisfactory for all situations, especially situations where coalitional behaviour by groups of players is likely. In the n-woman marriage game if the current state involves several women playing $TC$ and the man playing $CC$, then such a joint deviation could involve the man and one of the women agreeing to play $SC$. The man would clearly benefit from this deviation and the woman would also benefit as she would obtain the marriage contract with probability 1 rather than with probability $\frac{1}{n}$.

Consider the 3-woman marriage game if the situation were reversed, with profitable two player joint mutations occurring with probability of order $\epsilon$ and unprofitable single player mutations occurring with probability of order $\epsilon^2$. This effectively doubles the resistances for all the transitions where the payoff of the man decreases and the unique stochastically stable equilibrium in this case as $\epsilon \to 0$ is then $\sigma$. If we make random errors even more unlikely than joint mutations by increasing the power of epsilon we eventually get $\sigma$ as the unique stochastically stable state. That is, when rational coalitional behaviour is given sufficient priority over irrational single player deviations, the market power given to the man by there being more than one woman in the market is sufficient to take him to his favoured equilibrium (nb. In the above game this happens by a kind of Bertrand argument whereby if the man is not at his optimal contract the man and a woman deviate to a contract where they both do better in the short run and the other women follow so as to earn non-zero payoffs). It is also possible in some games that equilibrium selection can be altered by giving both of the above types of deviation probabilities of the same order (although it doesn’t affect the equilibrium chosen in the game above). It should also be noted that in the formulation of Young (1998a) it is always the players who stand to lose from a change of contract who induce the change through their making random errors. Naidu et al. (2010) address this by modelling ‘experimentation’ as a restriction which only allows changes in conventional contracts to be induced by random errors on the part of those who stand to gain from the change. We achieve the same without altering the myopic nature of the model by allowing both parties to gain in the short term from a joint deviation.
5 Coalitional stochastic stability

In this section we introduce our concept of coalitional stochastic stability. Like standard stochastic stability it is a concept based on limits and as such shares the benefits of its sharp predictive precision. It also shares the drawback of relying on the fact that the perturbations in question, — irrationality for SS, coalitional behaviour for CSS — disappear in the limit. However there are some important properties of CSS that SS does not share:

- CSS does not rely on payoff destructive behaviour by agents to gain sharp predictions.
- Versions of CSS can lead to much faster switching between absorbing states of the underlying unperturbed dynamic (discussed in detail later on in the paper).

It should be noted that the version of CSS below does include random errors as a technical tool to ensure irreducibility of the Markov process. There are many games where this will make no difference to predictions, and where it does, CSS without random errors will give sharp predictions for large classes of states - precisely those classes of states which are closed under rational coalitional behaviour.

First define the following notation.

$N$ is the set of all players. $|N| = n$.

$\mathcal{P}$ is the set of all subsets of $N$.

$\mathcal{P}_m \subset \mathcal{P}$ is the set of all subsets $P_m$ of $N$ such that $|P_m| = m$.

Let $F_m$ be a probability distribution over $\mathcal{P}_m$ with full support.

Given a state $h_t$, samples $\hat{p}_{t-1}^i$, $a_i \in B_t(\hat{p}_{t-1}^i)$ $\forall i$, and a set $Q \subset N$, let

$$A_Q(h_t) = \left\{ x : x_i \in B_t(\hat{p}_{t-1}^i) \ \forall i \notin Q, \ E[\pi_i(x_i)|x_j, j \in Q; \hat{p}_{t-1}^i] \geq E[\pi_i(a_i)|\hat{p}_{t-1}^i] \ \forall i \in Q \right\}.$$
$A_Q(h_t)$ is the set of action profiles where all players $i \notin Q$ play best responses to their sample distributions as in standard adaptive play and the players in $Q$ play actions such that the expected payoff of a player $j \in Q$ conditional on his knowing the actions of the other players in $Q$ and given his sample distribution of the actions of players outside $Q$ is higher than his expected payoff in standard adaptive play.

Let $G(A_Q(h_t))$ define a probability distribution over $A_Q(h_t)$ with full support.

Let $\hat{H}$ be a probability distribution with full support over $N$ and let $H_i$ be distributions with full support over all possible actions of player $i$.

Consider the following perturbed adaptive process $P^\epsilon$ with $\epsilon_i > 0 \forall i$

- With probability $1 - \sum_{i=2}^{n+1} \epsilon_i$ players follow the adaptive learning process as usual.

- With probability $\epsilon_m$ there is a Pareto superior deviation by $1 < m \leq n$ players. To be precise, a set of players $P_m$ is selected according to $F_m$. All $i \in N \setminus P_m$ play best responses to their sample distributions as normal. The actions of $i \in P_m$ change so that payoffs for these players are weakly better under the new action profile than they would be under individual best responses: given that the state in the previous period was $h_{t-1}$, a new action profile $x_t$ is selected from the distribution $G(A_{P_m}(h_{t-1}))$ and played in the current period. We call these deviations rational coalitional deviations of order $m$.

- With probability $\epsilon_{n+1}$ a random error occurs to the strategy of a randomly selected player. A player is selected according to $\hat{H}$ and he plays an action determined by $H_i$. All other players play best responses to their sample distributions as normal.

As the process is irreducible it has a unique stationary distribution which I denote (à la Young (1998b)) as $\mu^\epsilon$. A state $z$ is coalitionally stochastically
stable if:

\[ \lim_{\epsilon_2 \to 0} \lim_{\epsilon_3 \to 0} \lim_{\epsilon_4 \to 0} \ldots \lim_{\epsilon_n \to 0} \lim_{\epsilon_{n+1} \to 0} \mu^\epsilon(z) > 0. \]

5.1 What does this mean?

Effectively I have ranked the different types of deviations in order of importance. Most important are profitable single player deviations, followed by profitable two player deviations and so on. Least important of all are random unprofitable deviations. This order of importance is given by the order in which limits are taken. In working out our CSS states we take in turn:

- Any type of rational coalitional deviation to be infinitely more likely than unprofitable deviations.
- Rational coalitional deviations involving fewer players to be infinitely more likely than rational coalitional deviations involving more players.

So it is apparent that this concept puts a high value on rationality relative to the approach of Young (1993). Conversely it puts lower emphasis on the independence of the players’ actions. It can be understood to model a situation where the players in each position in the game are drawn each period from a population and, with some knowledge of how the game has been played in the past, best respond to how they expect their opponents to play, with the additional possibility that every so often groups of players will get together and adopt strategies to their mutual benefit.

5.2 Alternative approach

Another formulation of CSS might give the probability of a deviation by a coalition of size \( m \) a probability of \( \epsilon^m \) rather than of \( \epsilon_m \) and then proceed to take a single limit:

\[ \lim_{\epsilon \to 0} \mu^\epsilon(z) \]
The $b_m$ are then of cardinal and not just ordinal importance even in the limit. As an example consider the case when $b_2 = 2$ and $b_3 = 3$. Then a 3-player deviation still becomes infinitely less likely than a 2-player deviation as the limit is approached. However, it will always be the case - even in the limit, that three 2-player deviations will occur with exactly the same probability as two 3-player deviations.

If we let $\beta = \min_m \left( \frac{b_{m+1}}{b_m} \right)$ then it is not difficult to show that for a given underlying game $\Gamma$ there exists a value $\hat{\beta}$ such that for all $\beta > \hat{\beta}$, exactly the same states will be selected under this formulation as under our definition involving multiple limits. The benefits of CSS discussed in later sections also exist under this alternative formulation, but for ease and clarity of exposition we proceed with CSS as defined by multiple limits.

6 Results and examples

6.1 Fundamental propositions

The following propositions establish the existence of CSS states, give a method for finding them, and link them to the Nash equilibrium concept.\(^8\)

Proposition 1. CSS states exist and are identical to those selected by the following process:

- Take the recurrent classes under adaptive play with rational coalitional deviations of order $2, \ldots, n$. Find the recurrent class(es) with the lowest stochastic potential with respect to random unprofitable deviations (i.e. those with probability of order $\epsilon_{n+1}$).

- Within this recurrent class(es) take the recurrent classes under adaptive play with rational coalitional deviations of order $2, \ldots, n - 1$. Find which of these have the lowest stochastic potential with respect to coalitional deviations of order $n$ (i.e. those with probability of order $\epsilon_n$). Select these recurrent classes.

\(^8\)See appendix for proof of these and all subsequent propositions.
• Repeat with classes of order \( n - 2 \) and deviations of order \( n - 1 \).

• Keep going until you have selected recurrent classes of order 1.

• These are the coalitionally stochastically stable states of the underlying game. They comprise a subset of the recurrent classes of the process \( P^0 \) and, if singletons, are Nash Equilibria of the underlying game.

Proposition 2. Under conditions which in Young (1993) guarantee the selection of a convention(s) by SS, CSS will also select a convention(s). These conditions are:

• The underlying game \( \Gamma \) is weakly acyclic.

• The sample size \( s \) is sufficiently small relative to \( m \).

6.2 Example

\[
\begin{array}{cccccc}
 & L & M & N & O & R \\
 a & 6,2 & 4,1 & 30,0 & 0,\delta & 0,\delta \\
b & 1,8 & 3,7 & 0,0 & 0,\delta & 0,\delta \\
c & 0,\delta & 0,\delta & 0,0 & 7,3 & 1,5 \\
d & 0,\delta & 0,\delta & 0,0 & 8,1 & 2,6 \\
\end{array}
\]

Figure 3: A two player strategic game. \( \delta \) is assumed to be positive and close to zero.

Here we demonstrate how the algorithm in Proposition 1 works. \( \delta \) is assumed to be positive and very close to zero and is included so that action \( N \) is strictly dominated for player 2. The 4x5 game in Figure 3 has 2 strict Nash equilibria, \( aL \) and \( dR \), each of which corresponds to a convention of our unperturbed dynamic. Under the standard random errors approach, convention \( aL \) will then be selected as it takes relatively few errors by player 2 where he chooses \( N \) (a strictly dominated action for him) before it becomes worthwhile for player 1 to play \( a \) in the hope of earning a very big payoff at \( aN \). Once \( a \) is being played by player 1 it then becomes a best response
for player 2 to play $L$ and the convention $aL$ is reached. Under CSS it is clear that there is only one recurrent class under coalitional deviations of 2 players or fewer (i.e. those occurring with probabilities of order $\geq \epsilon_2$) and that this recurrent class includes both conventions: from convention $dR$ enough two player deviations to $bM$ will allow $aL$ to be reached, and from convention $aL$ enough two player deviations to $cO$ will allow $dR$ to be reached. As there exists only one recurrent class under coalitional deviations of 2 players or fewer there is no need to use random errors to choose between such recurrent classes. Within this recurrent class there are two recurrent classes under coalitional deviations of 1 player or fewer (i.e. those occurring with probabilities of order 1) and these recurrent classes are the conventions $aL$ and $dR$. We now use deviations of order 2 to choose between these conventions. From $dR$ the only possible two player coalitional deviation is to $bM$ and $\frac{2}{3}$ of these are required before it is possible that player 1 sampling from player 2’s actions sees player 2 playing $M$ often enough for player 1 to judge it worth his while to play $a$. If player 1 then plays $a$ for a while, player 2 can begin to play $L$ and $aL$ is reached. On the other hand, to move from $aL$ to $dR$ via plays of $cO$ requires only $\frac{2}{3}$ joint deviations before player 2 can switch to playing $R$ followed by player 1 switching to $d$. So our algorithm selects $dR$ as the unique coalitionally stochastically stable convention.

We have chosen between two conventions without either player engaging in behaviour that is detrimental to his short term myopic best interest. Irrational behaviour is not always necessary in order for stochastic stability arguments to have bite.

6.3 2×2 Games

Two player games are a special case when it comes to examining coalitional behaviour because any coalitional move in a 2 player game is by definition a move towards efficiency. In fact, if $\Gamma$ is a 2 player normal form game with multiple strict Nash Equilibria then CSS will never select an equilibrium which is Pareto inferior to another equilibrium which is itself not weakly dominated by another cell of the payoff matrix. For $2 \times 2$ games this leads to
the following result.

**Proposition 3.** When $\Gamma$ is a $2 \times 2$ game with more than a single strict Nash Equilibrium, CSS induces the following lexicographic decision rule:

(i) Where one equilibrium is Pareto superior to the other, the superior equilibrium is selected.

(ii) Where neither equilibrium is Pareto superior to the other, the risk dominant equilibrium is selected.

The question naturally arises as to how far we can extend our results regarding this preference of CSS for efficiency.

### 6.4 Efficiency with >2 players

In the example of the n-woman marriage game the selected equilibrium payoff vector is \((5, \frac{1}{n}, \ldots, \frac{1}{n})\). This is an element of the core of the game, defined as in Aumann and Peleg (1960), whether the core for games with non-transferable utility is described using the concept of $\alpha$-efficiency or $\beta$-efficiency.\(^9\) It is a well known property of the core that its elements are efficient outcomes of the underlying game. Can we establish any kind of inclusion relation between CSS and the core? The answer is no, we cannot. Even when CSS selects a unique equilibrium we cannot guarantee that this equilibrium is contained in the core of the game. This result is in contrast to Konishi and Ray (2003) where a farsighted dynamic process always selects payoffs in the core of the game when a unique limit of the process exists. Serrano and Volij (2005) demonstrate that stochastic stability does not necessarily select equilibria in the core.\(^{10}\)

Here I give an example of a game with a nonempty singleton core and a singleton CSS set which are not the same.

---

\(^{9}\) $\alpha$-efficiency guarantees coalitions payoffs at least as high as their maximin payoffs, $\beta$-efficiency guarantees coalitions payoffs at least as high as their minimax payoffs. The core under $\alpha$-efficiency is the set \{\((5, \frac{1}{n}, \ldots, \frac{1}{n}), (3, \frac{3}{n}, \ldots, \frac{3}{n}), (1, \frac{5}{n}, \ldots, \frac{5}{n})\)\}; the core under $\beta$-efficiency is \{\((5, \frac{1}{n}, \ldots, \frac{1}{n})\)\}.

\(^{10}\) This is clearly true for games with non-transferable utility. There is however more reason to suspect that CSS and the core might be related, both being defined using coalitional concepts.
In the game in Figure 4 the unique element of the core is $aLA$ with payoffs $(4, 4, 4)$. CSS chooses $bRB$ - the only inefficient pure strategy combination possible! The reason for this is that it requires a 3-player coalition to move from $bRB$ to $aLA$, whereas a 2-player coalition will deviate from $aLA$ to $bRA$ from where a best response of player 3 is to move to $bRB$ (all usual provisos about sample sizes in the adaptive process apply).

### 6.5 TU-Games

As TU-Games do not have a definitive 1-1 correspondence with non-cooperative games, the question arises as to how the concepts in this paper can be applied to TU-Games. Newton (2009) analyzes TU-Games interpreted as Nash demand games with some additional structure given by a characteristic function. It is shown that in such games the recurrent classes under adaptive play with rational coalitional deviations of order $2, \ldots, n$ correspond to the interior core allocations of the underlying TU-Game. This guarantees that CSS states lie in the interior core. The stochastic stability results of Newton (2009) can be interpreted as a characterization of the CSS states of such games under some additional conditions on the players’ samples and the process of strategic switching.

### 6.6 Coalition proofness

Nor is there an inclusion relation between CSS outcomes and Coalition Proof outcomes (Bernheim et al., 1987). The game in Figure 4 has $bRB$ as its unique CSS outcome and $aLA$ as its unique Coalition Proof outcome.

$aLA$ is coalition proof as all coalitional deviations lead to further devia-
tions by subsets of the deviating players.\footnote{Deviation to $bRA$ leads to deviation to $bLA$.} $bRB$ is not coalition proof as the players can jointly deviate to $aLA$ which is itself coalition proof. However, although a deviation from the Coalition Proof equilibrium $aLA$ to $bRA$ is disturbed by further deviations, it still allows the possibility of a transition to $bRB$ by single player best responses. Thus, coalitions that are not viable deviations in the Coalition Proof equilibrium concept can change outcomes in the CSS concept if they open up opportunities to enter the basin of attraction of another equilibrium.

The myopia of the players in this paper means that coalitional deviations are feasible even if they are not themselves robust to further coalitional deviations. Interpreted from the point of view of a population model, players selected from the population to play the game in question do not worry about the potential effect of their actions on future players in the same position in the game. As a consequence, CSS deals with a larger class of coalitional deviations than does Coalition Proof equilibrium, since the latter concept does not require robustness to coalitional deviations which can themselves be destabilized by further deviations by subsets of the set of deviating players. If we added a restriction to the process underlying CSS that restricted coalitional deviations to only those which were themselves coalition proof then we would have a concept of Coalition Proof Stochastic Stability (CPSS). Any stochastically stable Coalition Proof equilibrium would then be guaranteed to be CPSS, although the converse would not hold as Coalition Proof equilibria do not always exist whereas CPSS states do.

### 6.7 Structure of coalitional deviations

In my definition of CSS I assume that all possible coalitions have the chance to deviate. This can easily be altered to model situations where certain players are not expected to cooperate with one another. An example of this might be a game with a set of buyers and a set of sellers, where sets of sellers can make coalitional deviations (modelling collusive behaviour) but no other set of players can. We can in fact define any hierarchy of subsets of players...
ordered by the likelihood of the occurrence of coalitional behaviour in them and thus by the order in which limits will be taken to select CSS states:

\[ \xi_1, \xi_2, \ldots, \xi_M, \quad M \in \mathbb{N}, \; \xi_i \subset \mathcal{P} \forall i \]

Naturally, some coalition structures can be considered more reasonable than others. Suggestions have been made in the cooperative game theory literature that subsets of coalitions which are allowed to deviate should also be allowed to deviate\textsuperscript{12, 13} or alternatively that the union of coalitions which are allowed to deviate and have a nonempty intersection should also be allowed to deviate\textsuperscript{14}. In the first case it is argued that it is possible for a subset of a set of players who meet to discuss strategy to meet without the others present. The second case is predicated on the argument that players who are members of the intersection between two coalitions can serve as intermediaries to bring the interests of the two coalitions together.

### 6.8 Example: Bertrand game with convex costs

Consider a model of Bertrand competition with a single undifferentiated good. There are \( n \) firms producing the good. Firms simultaneously set prices \( p \in \mathbb{N} \textsuperscript{15} \) and the lowest price seller gets all of the demand with demand being shared equally if there are two or more firms with equally low prices. Total demand for the good is \( D \) and the cost to a firm of producing \( d \) units of the good is given by \( ad + bd^2 \) where \( a \) and \( b \) are strictly positive constants. For simplicity we assume that \( D \) is not affected by the price \( p \).

Clearly, all strict Nash Equilibria of this game involve every firm charging the same price. Some of these Nash Equilibria are vulnerable to subsets of

\textsuperscript{12}Of course the argument is not phrased in this manner in cooperative papers. Instead the structure of allowable deviations in cooperative games is described by the set of characteristic function inequalities that need to be satisfied in order for a game to be counted as part of the core of the game or to satisfy another cooperative solution concept such as the nucleolus (Schmeidler (1969))

\textsuperscript{13}Algaba et al. (2000)

\textsuperscript{14}Algaba et al. (1999)

\textsuperscript{15}Note that although here we use discrete prices, the arguments of this section are equally valid for the continuous pricing Bertrand model.
players dropping their prices by 1 and sharing the market between themselves; in fact there is a threshold price \( p' \) above which it is worthwhile for \( k \) players to do just that. This threshold is given by:

\[
\frac{D}{n} p' - a \frac{D}{n} - b \left( \frac{D}{n} \right)^2 = \frac{D}{k} (p' - 1) - a \frac{D}{k} - b \left( \frac{D}{k} \right)^2
\]

which gives:

\[
p' = a + b D \frac{n}{n} + b D \frac{k}{n - k}
\]

which is minimized over \( k \) at:

\[
k^* = \frac{n \sqrt{bD}}{\sqrt{n} + \sqrt{bD}}
\]

giving:

\[
p^* = a + b D + (b \sqrt{bD} + \sqrt{n})(\sqrt{bD} + \sqrt{n}) \frac{n}{n}
\]

So if, for example, \( n = 10 \), \( a = 10 \), \( b = 0.1 \), \( D = 100 \), then price equals average cost at \( p = a + b D \frac{n}{n} = 11 \) and \( p'(1) = 22.11 \), suggesting that any price between 11 and 22 sustains a Nash Equilibrium. However, \( k^* = 5 \) and \( p^* = 15 \), so every Nash Equilibrium at prices \( p \geq 15 \) is vulnerable to deviations by coalitions of 5 players. The set of strategies can be made finite by only looking at prices up to 25 and then this game can be analyzed in an adaptive play setting. CSS then selects a convention as follows: the unique recurrent class of states under coalitional deviations of 10 players or fewer includes the states where all firms set the same price between 12 and 25 inclusive. There are three recurrent classes of states under coalitional deviations of 9 players or fewer: the conventions where all firms set prices 12, 13 and 14 respectively. The easiest way out of these recurrent classes by 10-player deviations is when the deviations in question involve all firms setting \( p = 25 \). After this has occurred enough times it is then possible for a 9-player deviation to better respond with \( p = 24 \). This transition out of
the recurrent class will be harder the higher the price being charged to begin with, therefore the convention where all firms set price \( p = 14 \) is the unique CSS state.\(^{16}\)

As the number of firms in the market becomes large \( p'(1) \rightarrow a + bD + 1 \). In the example above this means \( p'(1) \rightarrow 21 \) which suggests that the set of Nash Equilibria is not significantly reduced in size. However, \( p^* \rightarrow a + 1 \): as the market becomes competitive the only Nash Equilibrium robust to coalitional deviations of \((n - 1)\) players or fewer and hence the only CSS equilibrium is that where \( p = a + 1 \). Moreover, although \( k^* \rightarrow \infty \), \( \frac{k^*}{n} \rightarrow 0 \): the number of firms involved in the coalitional deviation which persists at the lowest prices increases, but that number as a share of the total number of firms in the market decreases.

### 6.9 Experimentation

It is mentioned above that CSS gives added weight to the explanation of the small probability events underlying SS as experimentation. When there is a tiny chance (the chance that another player simultaneously randomly experiments) of ‘experimentation’ leading to increased payoffs it seems a funny kind of behaviour for players to engage in. For this reason, justifications usually involve arguments that depart from the myopia of standard adaptive learning. However, with coalitional behaviour it becomes possible for there to be a non-negligible possibility of ‘experimentation’ resulting in increased payoffs for the players concerned. For completeness we outline a slightly different model to our one above which better fits this interpretation.

Consider the following perturbed adaptive process \( E^\epsilon \) with \( \epsilon_i > 0 \ \forall i \)

- There is an action profile being played at the start of the period.

- With probability \( 1 - \sum_{i=1}^{n+1} \epsilon_i \) players best respond to the existing action profile.

\(^{16}\)In \( n \)-firm differentiated product oligopoly models such as that of Kandori and Rob (1995) where a firm’s best response only depends on the average of its opponents’ strategies and there are multiple strict symmetric Nash equilibria, similar arguments can be used to show that the equilibrium which gives the highest payoffs to the firms will be selected.
• With probability $\epsilon_m$ $m$ players are selected randomly and choose random actions. They keep these actions if they all do at least as well under the new action profile as under the existing action profile. Otherwise they revert to the actions they were playing at the start of the period.

So we have a model of experimentation to which stochastic stability arguments can be applied which does not rely on irrational short term behaviour by players.

6.10 Time to convergence

One of the greatest problems affecting stochastic stability arguments is the large length of time it can take on average to move to a stochastically stable convention from a non-SS convention. Consider the following game:

$$\begin{array}{c|cc}
\text{Player 1} & 1 & 2 \\
\hline
1 & 10,10 & 0,0 \\
2 & 0,0 & 8,8 \\
\end{array}$$

If $m = 15$, $s = 6$ then three mutations by either player are required to move from the convention $(2, 2)$ to the convention $(1, 1)$. If $\epsilon = \frac{1}{10}$ we can expect this to happen over any given three periods with probability of order $\epsilon^3 = \frac{1}{1000}$. That is, a lot of time is expected to elapse before a move to $(1, 1)$.

In fact, starting from $(2, 2)$ it takes an average of close to 1708 periods to make the transition to $(1, 1)$. With CSS however, it is not too hard to imagine that following a coalitional deviation to $(1, 1)$ further coalitional deviations might be more likely or even certain. This would trivially cut the expected time before a move to $(1, 1)$ occurs. Such an assumption makes sense because even if the ‘players’ are emerging from populations of agents, you would expect such successful behaviour to have some chance of being communicated and thus replicated in the following period. This argument cannot be used with standard SS: in fact under SS you might expect errors to become even more unlikely following an error given the damaging effect of errors on immediate
Table 1: Mean transition times in periods (standard errors under 1.1% of value). Coalitional behaviour gives faster convergence. Probability of deviation of $\frac{1}{10}$, except with stickiness where in the period immediately following a deviation the probability of the same deviation recurring is $\frac{1}{2}$. Transition times are the number of periods elapsed from an initial state where (2, 2) is the convention until the first transition to the convention (1, 1) occurs.

<table>
<thead>
<tr>
<th>Random errors</th>
<th>Coalitional</th>
</tr>
</thead>
<tbody>
<tr>
<td>No stickiness</td>
<td>Stickiness</td>
</tr>
<tr>
<td>1708</td>
<td>382</td>
</tr>
<tr>
<td></td>
<td>98</td>
</tr>
</tbody>
</table>

payoffs. Numerically, if we set $\epsilon_2 = \frac{1}{10}$ and the probability of a random error to zero then we get a faster average transition time of 382 periods due to both players changing their actions with each deviation. Furthermore, if following a deviation we add stickiness in that a deviation has a probability of $\frac{1}{2}$ of being repeated in the next period, then we get a much faster average transition time of 98 periods. Results are summarized in table 1.

There is also a second reason why coalitional considerations may give faster convergence times than random error formulations. To see why this is we consider the game in Figure 5. This game is such that a given player will always choose the action which gives him the highest probability of coordinating with the other player. However, when a player thinks his opponent equally likely to play any of several actions he will try for the pareto superior coordination outcome. Both SS and CSS choose (1, 1) as the unique stable outcome. With $\epsilon = \frac{1}{10}$, we ask how long it takes to reach (1, 1) from (2, 2). Under a uniform error model, the additional actions available increase the transition time: with many possible errors to make, the chance of players repeatedly making the same error is reduced and the expected transition time increases. This is easy to miss as it is irrelevant in the limit as the probability of an error occurring approaches zero. The element of rationality underlying coalitional deviations precludes the problem: two players making a coalitional deviation will never deviate to one of the equilibria of the underlying game which gives them a worse payoff than the convention at which they find themselves. Predictions made with coalitional considerations in mind are therefore more robust to (intuitively irrelevant) changes in the game.
Table 2: Mean transition times in periods (standard errors under 1.1% of value). Coalitional behaviour gives faster convergence. Probability of deviation of \( \frac{1}{10} \). All possible pareto superior coalitional deviations are given equal probability. With \textit{stickiness}, in the period immediately following a deviation the probability of the same deviation recurring is \( \frac{1}{2} \).

Results are summarized in table 2.

\[
\begin{array}{cccc}
1 & 2 & \cdots & k \\
1 - \delta, 1 - \delta & 0, 0 & \cdots & 0, 0 \\
0, 0 & 1 - 2\delta, 1 - 2\delta & \cdots & 0, 0 \\
\vdots & \vdots & \ddots & \vdots \\
0, 0 & 0, 0 & 0, 0 & 1 - k\delta, 1 - k\delta
\end{array}
\]

Figure 5: More available actions can increase convergence times. \( \delta \) is assumed to be positive and close to zero.

We stress that these arguments are not opposed to those of Young (1993) but rather complementary to them. If we understand the model as being that of the same game being played repeatedly between representatives of different populations then the assumptions of uniform errors and no coalitional behaviour become more realistic the larger the underlying populations. The arguments giving CSS faster convergence times are stronger the smaller the underlying population and the better the communication of successful strategies within such populations. This brings us to the next section, which takes a look at an area where standard stochastic stability is (we argue) an often inappropriate but frequently used tool.
6.11 Local interaction

"Local matching rules are appropriate to describe situations where players interact not with the population as a whole, but rather with a few close friends or colleagues. For example, such a rule might describe the interactions at a college reunion where each participant knows in advance who he or she wishes to see." Ellison (1993)

Stochastic stability has been used in a variety of local interaction models, for example Ellison (1993), Ellison (2000), Eshel et al. (1998), Goyal and Vega-Redondo (1999). Ellison (1993) notes that convergence to stochastically stable states can be much faster when players interact with a small group of neighbours than when they are uniformly matched across an entire population. Ellison notes that it is possible for models to have convergence in reasonable time and gives an example where this is indeed the case: local interaction models of coordination where each player can choose between two strategies $A$ and $B$ and wishes to play strategy $A$ if and only if at least a certain proportion of his neighbours are also playing strategy $A$.

As an immediate caveat to the above observation we would like to note that time to convergence in this type of model can depend enormously on the number of strategies available to the players concerned. This is similar to the arguments made in the preceding section and again is easy to miss because the effect disappears in the limit as the probability of an error occurring approaches zero. Consider 5 players who have all committed errors (for a given set of 5 players this occurs with probability $\epsilon^5$ in a uniform error model). Let us say that when acting in error a player has no preference over the action he plays. Then if players have 2 actions available to them, they will all play the same action with probability $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. However, if they have 10 actions available they will all play the same action with probability $\left(\frac{1}{10}\right)^4 = \frac{1}{10000}$. It is clear from this that the number of actions available can have a large effect on convergence times as helpful coordination of errors becomes less likely. Again, it can be immediately seen that this problem does not arise in CSS, where although coalitional behaviour may be a small
probability event, when it does take place coordination will be automatic. Applying the structure of local interaction models to coalitional behaviour is also easy: we simply take coalitional behaviour by players who are not ‘close’ to one another to be impossible. Table 3 outlines the reasons why we find coalitional arguments superior to standard SS in local interaction models.

<table>
<thead>
<tr>
<th>SS</th>
<th>CSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Movement between conventions depends on a degree of unpredictability in the actions of a player’s neighbours, who he supposedly knows reasonably well.</td>
<td>Actions leading to movement between conventions are determined by rational behaviour jointly agreed by players who know one another.</td>
</tr>
<tr>
<td>Spread of new ideas and conventions is an essentially random process.</td>
<td>Network effects and the spread of ideas are modelled in a rational way.</td>
</tr>
<tr>
<td>New technology modelled as being randomly adopted by multiple agents at once.</td>
<td>Can model decisions to introduce new technologies as being jointly made by those who stand to benefit.</td>
</tr>
</tbody>
</table>

Table 3: Local interaction: SS vs. CSS

However, coalitional behaviour does not always lead to faster convergence times, and can in fact lead to the opposite effect as groups of players adjust their strategies to maintain the current convention. Proposition 4 shows how different effects can predominate depending on the parameters of the game. Consider the model of Ellison played on a two dimensional lattice with no edges such as a lattice on a torus (Figure 6). Let ξ be the number of a player’s neighbours who play the same action as he does. Let payoffs to a player be π = ξ if he plays B and π = αξ if he plays A, 1 < α < 2. To simplify analysis set s = 1, m = 2. We compare the convergence times of SS with error probabilities ǫ to convergence times of CSS with ǫ2 = ǫ1−η, ǫ3 = ǫ4 = ... = ǫn = 0, ǫn+1 = ǫ. We denote the convention where all players play A by CA, the convention where all players play B by CB (Figure 7), and the
convention where four players forming a square play $A$ and the rest play $B$ by $C_2$ (Figure 8). We denote the expected waiting time of a transition from state $x$ to state $y$ by $W_{SS}(x, y, \epsilon)$ and $W_{CSS}(x, y, \epsilon)$ for SS and CSS processes respectively.

**Proposition 4.** Take Ellison’s model of local interaction on a 2-dimensional lattice adapted as above.

When $\alpha < 1.5$:

$$\lim_{\epsilon \to 0} \frac{W_{CSS}(C_B, C_A, \epsilon)}{W_{SS}(C_B, C_A, \epsilon)} = \infty$$
When $\alpha > 1.5$:

$$\lim_{\epsilon \to 0} \frac{W_{CSS}(C_B, C_A, \epsilon)}{W_{SS}(C_B, C_A, \epsilon)} \asymp 1, \quad \lim_{\epsilon \to 0} \frac{W_{CSS}(C_2, C_A, \epsilon)}{W_{SS}(C_2, C_A, \epsilon)} = 0$$

In both SS and CSS formulations, the least resistance path from $C_B$ to $C_A$ starts off with two random errors leading to $C_2$. If $\alpha < 1.5$ a player would rather play $B$ and have three neighbours playing $B$ than play $A$ and have two neighbours playing $A$, so from $C_2$, two adjacent players from the four playing $A$ could coalitionally deviate to playing $B$, taking the state back into the basin of attraction of $C_B$. This conservative effect of coalitional behaviour is what gives the first part of Proposition 4. If $\alpha > 1.5$ then the effect is reversed and from $C_2$ two player coalitions occurring with probability of order $\epsilon^{1-\eta}$ increase the size of the block of players playing $A$. This decreases the transition time from $C_2$ to $C_A$. However, as $\epsilon$ becomes small, by far the greatest component of the transition time from $C_B$ to $C_A$ comes from the two simultaneous errors required to move from $C_B$ to $C_2$.

## 7 Conclusion

This paper has presented the use of coalitional stochastic stability as a method of equilibrium selection, and argued that it should be preferred to random error based stochastic stability wherever coalitional behaviour is feasible. We have demonstrated that the ideas underlying CSS are as intuitive
as those underlying standard stochastic stability and shown how CSS states can be found. CSS is a way of incorporating coalitional considerations into equilibrium selection and evolutionary game theory and therefore lies at a crossroads of several different strands of literature. We have shown how despite a strong preference for efficiency in the description of CSS, efficiency will not always be attained and that sometimes social movements to other Nash equilibria (such as the French Revolution or the move from bRB to aLA in Figure 4) will quickly collapse due to further deviations. This does not mean that such changes will not happen. They will happen, and the resulting instability may take you somewhere new.

This paper has demonstrated that unlike standard stochastic stability approaches, a coalitional approach does not rely on payoff destructive behaviour by individual players, and this helps us to justify the interpretation of random behaviour as experimentation. The payoff beneficial nature of coalitional deviations also allows models to be built which converge much faster to the stable states than has been the case with previous models, reducing history dependence and enabling the modelling of social change on a more realistic timescale.

Coalitional behaviour is something that can be observed in many noncooperative games and in discussions of Nash and other equilibrium concepts it is often the elephant in the room: the question being how to incorporate the realism of coalitional behaviour without discarding the precision of equilibrium predictions. This paper has given one way of overcoming this problem. It would be interesting to see further analysis of games with coalitions and a multiplicity of equilibria using the tools described in this paper.

References


8 Appendix

Proof of propositions 1 and 2. Define $P^\epsilon \leq n$ as identical to $P^\epsilon$ in all respects other than that $\epsilon_{n+1} = 0$. Denoting the unique stationary distribution of $P^\epsilon$ as $\mu^\epsilon$ for each $\epsilon_{n+1} > 0$, we know from Theorem 4 of Young (1993) that $\lim_{\epsilon_{n+1} \to 0} \mu^\epsilon = \mu^\epsilon \leq n$ exists and $\mu^\epsilon \leq n$ is a stationary distribution of $P^\epsilon \leq n$. We also know that the states $z$ with $\mu^\epsilon \leq n > 0$ are contained in the recurrent classes of $P^\epsilon \leq n$ with the lowest stochastic potential. Take one of these recurrent classes and call it $Z$. The process $P^\epsilon \leq n$ restricted to $Z$ is clearly irreducible and positive recurrent. Hence it has a unique stationary distribution which must be $\mu^\epsilon \leq n$ restricted to the states in $Z$ and scaled so as to sum to 1. We denote this distribution $\mu^\epsilon \leq n_Z$. Define $P^\epsilon \leq n - 1$ as identical to $P^\epsilon \leq n$ except that $\epsilon_n = 0$. Define $P^\epsilon \leq n_Z$ and $P^\epsilon \leq n - 1_Z$ as these processes restricted to $Z$. Then reiterating Young’s Theorem we have that $\lim_{\epsilon_n \to 0} \mu^\epsilon \leq n_Z = \mu^\epsilon \leq n - 1_Z$ exists and is a stationary distribution of $P^\epsilon \leq n - 1$. As this applies to every possible recurrent class $Z$ of $P^\epsilon \leq n$ we then have that $\lim_{\epsilon_n \to 0} \mu^\epsilon \leq n = \mu^\epsilon \leq n - 1$ exists and is a stationary distribution of $P^\epsilon \leq n - 1$. Now if we regard deviations of $n$-players occurring with probability of order $\epsilon_n$ as the deviations used to measure stochastic potential, it is immediately clear that for any $Z$, the states $z \in Z$ such that $\mu^\epsilon \leq n - 1_Z > 0$ are the states with the lowest stochastic potential.

Continuing in this fashion we see that

$$\lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \lim_{\epsilon_4 \to 0} \lim_{\epsilon_n \to 0} \lim_{\epsilon_{n+1} \to 0} \mu^\epsilon(z) = \mu^0(z)$$

exists and is a stationary distribution of $P^0$. It is clear from the above that the states $z$ with $\mu^0(z) > 0$ can be calculated using the process described in Proposition 1. Given that CSS selects a stationary distribution of $P^0$ and that we know from Young (1993) that stationary distributions of $P^0$ select convention(s) under the conditions given in Proposition 2, the proof is complete.

□

Proof of proposition 3. A $2 \times 2$ game with two strict NE is clearly weakly
acyclic, so Proposition 2 is satisfied and a convention will be chosen. If one equilibrium is Pareto superior to the other it is the unique recurrent class of \( P^{\leq 2} \) and so is selected by CSS. If neither NE is Pareto superior to the other both of them are singleton recurrent classes in \( P^{\leq 2} \). Clearly whichever of them is selected by the random errors of order \( \epsilon_3 \) will be the selected convention. Effectively we are selecting our CSS states only using the random errors and the selection is reduced to the standard stochastic stability notion which Young (1993) shows selects the risk dominant equilibrium in \( 2 \times 2 \) games.

Proof of proposition 4.

When \( \alpha < 1.5 \):

For CSS the lowest cost way out of \( C_B \) is for 2 random errors to lead to \( C_2 \). The expected waiting time for this transition is proportional to \( \epsilon^{-2} \). As the transition \( C_2 \rightarrow C_B \) can be caused by \( \epsilon^{1-\eta} \) probability events and the next lowest probability way to exit \( C_2 \) is via a \( \epsilon \) probability random error, the expected number of repetitions of \( C_B \rightarrow C_2 \) before \( C_A \) is attained is approximately proportional to \( \frac{\epsilon^{1-\eta}}{\epsilon} \). Therefore \( W_{CSS}(C_B, C_A, \epsilon) \) is at least of order \( \epsilon^{-2} \frac{\epsilon^{1-\eta}}{\epsilon} = \epsilon^{-2-\eta} \). There is a possible transition path (Figure 9) from \( C_B \) to \( C_A \) that goes \( C_B \rightarrow C_2 \rightarrow C_3 \rightarrow \ldots \rightarrow C_A \) where \( C_3 \) is a 3 by 3 block of players playing \( A \) and each step in the transition apart from the first involves a single \( \epsilon \) probability error leading to a larger rectangular block of players playing \( A \) until \( C_A \) is reached. Each of these transitions is the least cost transition away from the state in question apart from the transition from \( C_2 \) which is more costly than the \( \epsilon^{1-\eta} \) transition back to \( C_B \). So by Theorem 2 of Ellison (2000), \( W_{CSS}(C_B, C_A, \epsilon) = O(\epsilon^{-2-\eta}) \). Therefore \( W_{CSS}(C_B, C_A, \epsilon) \) is asymptotically proportional to \( \epsilon^{-2-\eta} \).

The situation for SS (Figure 10) is identical to that for CSS, apart from the fact that the easiest transition \( C_2 \rightarrow C_B \) now requires a single random error of probability \( \epsilon \). This gives us the result that \( W_{SS}(C_B, C_A, \epsilon) \) is asymptotically proportional to \( \epsilon^{-2} \). So, for some constant \( \phi \):

\[
\lim_{\epsilon \to 0} \frac{W_{CSS}(C_B, C_A, \epsilon)}{W_{SS}(C_B, C_A, \epsilon)} = \lim_{\epsilon \to 0} \phi \frac{\epsilon^{-2-\eta}}{\epsilon^{-2}} = \lim_{\epsilon \to 0} \phi \epsilon^{-\eta} = \infty
\]
When \( \alpha > 1.5 \):

The lowest cost way out of \( C_B \) is for 2 random errors to lead to \( C_2 \). The expected waiting time for this transition is proportional to \( \epsilon^{-2} \). There is a possible transition path (Figure 11) from \( C_B \) to \( C_A \) that goes \( C_B \to C_2 \to C_{3,2} \to \ldots \to C_A \) where \( C_{3,2} \) is a 3 by 2 block of players playing \( A \) and each step in the transition apart from the first involves a single \( \epsilon^{1-\eta} \) probability 2-player deviation leading to a larger rectangular block of players playing \( A \) until \( C_A \) is reached. Each of these transitions is the least cost transition away from the state in question, so by Theorem 2 of Ellison (2000), \( W_{CSS}(C_B, C_A, \epsilon) = O(\epsilon^{-2}) \). Therefore \( W_{CSS}(C_B, C_A, \epsilon) \) is asymptotically proportional to \( \epsilon^{-2} \). The same reasoning applies for SS (Figure 12) only with \( \epsilon \) probability random errors taking the place of the 2-player coalitional deviations. Therefore \( W_{SS}(C_B, C_A, \epsilon) \) is asymptotically proportional to \( \epsilon^{-2} \) and for some constant \( \phi \):

\[
\lim_{\epsilon \to 0} \frac{W_{CSS}(C_B, C_A, \epsilon)}{W_{SS}(C_B, C_A, \epsilon)} = \lim_{\epsilon \to 0} \phi \frac{\epsilon^{-2}}{\epsilon^{-2}} = \phi
\]

The expected waiting times for the transition \( C_2 \to C_A \) depend on the probability of moving back to \( C_B \) as the waiting time that comes from escaping \( C_B \) comes to dominate as \( \epsilon \) becomes small. For CSS \( C_2 \to C_B \) requires a \( \epsilon \) probability random error, whereas \( C_2 \to C_3 \) requires only a \( \epsilon^{1-\eta} \) 2-player deviation. So for small \( \epsilon \) the probability of \( C_2 \to C_B \) occurring is asymptotically proportional to \( \epsilon^{1-\eta} = \epsilon^{\eta} \) and expected waiting time is asymptotically proportional to \( \epsilon^{\eta} \).

For SS both \( C_2 \to C_B \) and \( C_2 \to C_3 \) require \( \epsilon \) probability random errors so expected waiting time is approximately proportional to \( \epsilon^{-2} \). Therefore, for some constant \( \phi \):

\[
\lim_{\epsilon \to 0} \frac{W_{CSS}(C_2, C_A, \epsilon)}{W_{SS}(C_2, C_A, \epsilon)} = \lim_{\epsilon \to 0} \phi \frac{\epsilon^{\eta-2}}{\epsilon^{-2}} = \lim_{\epsilon \to 0} \phi \epsilon^{\eta} = 0
\]
\[ \begin{align*}
&\mathbf{C}_B \xrightarrow{\epsilon_2} \mathbf{C}_2 \xleftarrow{\epsilon_1^\eta} \mathbf{C}_3 \xrightarrow{\epsilon} \mathbf{C}_A \quad \text{Figure 9: } \alpha < 1.5, \text{ CSS transition } C_B \rightarrow C_A. \\
&\mathbf{C}_B \xrightarrow{\epsilon_2} \mathbf{C}_2 \xrightarrow{\epsilon} \mathbf{C}_3 \xrightarrow{\epsilon} \mathbf{C}_A \quad \text{Figure 10: } \alpha < 1.5, \text{ SS transition } C_B \rightarrow C_A. \\
&\mathbf{C}_B \xrightarrow{\epsilon_2} \mathbf{C}_2 \xrightarrow{\epsilon_1^\eta} \mathbf{C}_{3,2} \xrightarrow{\epsilon_1^\eta} \mathbf{C}_A \quad \text{Figure 11: } \alpha > 1.5, \text{ CSS transition } C_B \rightarrow C_A. \\
&\mathbf{C}_B \xrightarrow{\epsilon_2} \mathbf{C}_2 \xrightarrow{\epsilon} \mathbf{C}_{3,2} \xrightarrow{\epsilon} \mathbf{C}_A \quad \text{Figure 12: } \alpha > 1.5, \text{ SS transition } C_B \rightarrow C_A. 
\end{align*} \]
9 Appendix: Theorems cited in the text

Here are statements of a couple of theorems used in the proofs of this paper.

9.1 Theorem 4 (Young, 1993)

Let $P^0$ be a stationary Markov process on the finite state space $X$ with recurrent communication classes $Z_1, Z_2, \ldots, Z_J$. Let $P^\epsilon$ be a perturbation of $P^0$ satisfying:

1. $P^\epsilon$ is aperiodic and irreducible for all $\epsilon > 0$,
2. $\lim_{\epsilon \to 0} P^\epsilon_{xy} = P^0_{xy}$,
3. $P^\epsilon_{xy} > 0$ for some $\epsilon$ implies $\exists r \geq 0$ such that $0 < \lim_{\epsilon \to 0} \epsilon^{-r} P^\epsilon_{xy} < \infty$.

and let $\mu^\epsilon$ be its unique stationary distribution for every small positive $\epsilon$.

Then:

(i) As $\epsilon \to 0$, $\mu^\epsilon$ converges to a stationary distribution $\mu^0$ of $P^0$.

(ii) $\mu^0_z > 0$ if and only if $z$ is contained in a recurrent class $Z_j$ that minimizes stochastic potential.

9.2 Theorem 2 (Ellison, 2000)

Define the radius $R(Z_i)$ of $Z_i$ as the resistance of the easiest path from $Z_i$ to another recurrent class:

$$R(Z_i) = \min_{Z_j \neq i} r(Z_i, Z_j)$$

Defining the resistance of a path from $Z_1$ to $Z_T$ as:

$$r(Z_1, Z_2, \ldots, Z_T) = \sum_{i=1}^{T-1} r(Z_i, Z_{i+1})$$

a modified resistance can be calculated. This can be understood as adjusting resistances for how likely it is that after reaching a recurrent class $Z_i$, the
process will exit to a recurrent class other than $Z_{i+1}$.

$$r^*(Z_1, Z_2, \ldots, Z_T) = r(Z_1, Z_2, \ldots, Z_T) - \sum_{i=2}^{T-1} R(Z_i)$$

$$r^*(Z_1, Z_T) = \min_{(Z_1, Z_2, \ldots, Z_T)} r^*(Z_1, Z_2, \ldots, Z_T)$$

Following this, the \textit{modified coradius} $CR^*(Z_i)$ can be defined. This can be roughly understood to mean how easy it is to get to $Z_i$ from any starting state after resistances have been adjusted.

$$CR^*(Z_i) = \max_{Z_j \neq i} r^*(Z_j, Z_i)$$

Then if for some recurrent class $Z_i$, $R(Z_i) > CR^*(Z_i)$:

(i) The stochastically stable states of the model are contained in $Z_i$.

(ii) For any $Z_j \neq Z_i$, $W(Z_j, Z_i, \epsilon) = O(\epsilon^{-CR^*(Z_i)})$ as $\epsilon \to 0$. 

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