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Abstract

Gerard Debreu introduced a well known radial efficiency measure which he called a "coefficient of resource utilization." He derived this scalar from a much less well known "dead loss" function that characterizes the monetary value sacrificed to inefficiency, and which is to be minimized subject to a normalization condition. We use Debreu's loss function, together with a variety of normalization conditions, to generate several popular families of linear efficiency models. Linearity is a computational consideration rather than a theoretical requirement. Our methodology also can be employed to generate entirely new families of linear efficiency models.

Key words: DEA; Efficiency; LP

1. Introduction

The famous paper by Debreu [1], “The Coefficient of Resource Utilization,” has inspired this study. Farrell [2, pp. 253-54] remarked that “The professional economist...can note the similarity of the measure of ‘technical efficiency’ and Debreu’s ‘coefficient of resource utilization’,” although in our opinion the similarity has been exaggerated. However the concept of Debreu that has most influenced this study is his loss function, which Farrell did not mention, and which has gone largely overlooked in the literature¹. This concept, which was initially developed for evaluating the “dead loss” associated with a non-optimal allocation of resources in an economic system, is a money metric measure of the distance from an actual allocation to a set of optimal allocations, i.e., “the minimum of the distance from the given complex to a varying optimal complex.” After proving “the intrinsic existence of price systems associated with the optimal complexes of physical resources,” the minimization problem proposed by Debreu is

$$\text{Min}_z p_z \cdot (z_0 - z),$$

with z_0 a vector representing the actual allocation of resources, z a vector belonging to the set of optimal allocations and p_z is one of the corresponding *intrinsic* price vectors. Debreu named the optimal value of this problem “the magnitude of the loss”, and he proved that $p_z \cdot (z_0 - z) \geq 0$, realizing that “ p_z is affected by an arbitrary positive scalar”. The influence of this multiplicative scalar means that the magnitude of the loss can be driven to zero by an appropriate scaling of all elements of p_z . “In order to eliminate the arbitrary multiplicative factor affecting all the prices,” Debreu proposed to divide the

¹ As far as we know Diewert [3] extended Debreu’s loss measure, but in a different context and in a different way as we do. Diewert focused his analysis on measuring the loss of output that can be attributed to distortions within the production sector of an open economy. In addition, Diewert did not consider different normalization conditions as we do. More recently, ten Raa [4] has modified Debreu’s coefficient of resource utilization in the context of total factor productivity growth.

objective function by a price index, either $p_z \cdot z_0$ or $p_z \cdot z$, reformulating the original problem as

$$\text{Min}_z p_z \cdot (z_0 - z) / p_z \cdot z_0, \text{ or, equivalently, } \text{Max}_z p_z \cdot z / p_z \cdot z_0.$$

It is clever to show, as Debreu did, that an optimal solution to the maximization problem is $z^* = \rho z_0$, where the scalar $\rho \leq 1$ is Debreu's "coefficient of resource utilization." Moreover, $\rho = 1$ means that the actual allocation z_0 belongs to the set of optimal allocations (i.e., is efficient).

We observe that the minimization problem has as variables both p_z and z , and is nonlinear and hence difficult to solve. We stress that it is not compulsory to resort to a normalization factor because the "adverse" influence of the arbitrary multiplicative scalar can be eliminated by adding restrictions to the loss minimization problem. In fact, Debreu's problem can be rewritten as

$$\begin{array}{l} \text{Min}_z \quad p_z \cdot (z_0 - z) \\ \text{s.t.} \quad p_z \cdot z_0 = 1 \end{array}$$

We observe that neither the normalization condition nor the added restrictions are unique. Observe also that the normalization condition involves all the intrinsic prices, just as the normalization factors of Debreu do.

Debreu studied an economic system consisting of two activities, production and consumption, and having three sources of loss, underemployment of resources, inefficiency in production and imperfection of economic organization. We simplify matters by studying the production activity of an economic system having one source of loss, which Debreu calls "the technical inefficiency of production units." In a production context we can use the loss function minimization method introduced by Debreu to evaluate the technical efficiency of any producer, assuming that the optimal producers have intrinsic prices affected by a positive scalar unless a normalization scheme is introduced. In our

case, the existence of nonnegative intrinsic prices is guaranteed by the structure we impose on the production set. Moreover these assumptions also allow us to simplify our initially designed model by eliminating some of its variables.

The paper unfolds as follows. In section 2 we list the requirements that the production set must satisfy, and we formulate an initial version of our loss function minimization model. This version of the model is formulated in a generic way because the restrictions relating the set of intrinsic prices to the corresponding optimal allocation are not formulated mathematically, and so the normalization condition is not explicitly specified. This model seeks, similar to Debreu's method, the minimum of the distance from the production unit under evaluation to a varying optimal allocation in the production set, and depends both on the optimal allocation and on its intrinsic prices. We then obtain a second version of the model, equivalent to the initial version, which linearizes the loss function minimization model and characterizes the geometric nature of the model as a supporting hyperplane model. It does so by eliminating the optimal allocation from the minimization problem, which depends only on the set of intrinsic prices and the intercept of the supporting hyperplane. In section 3 we further specialize the second version of the model by introducing a common set of mathematical restrictions, but with a sequence of different normalization conditions, giving rise to several well known families of efficiency models that either are linear or can be linearized. In section 4 we examine the important property of units invariance of the linear loss function models we consider in section 3. We prove two propositions that enable us to characterize models as being units invariant or units invariant under certain conditions. Section 5 concludes.

2. The Loss Function Models

In this section the loss function is defined in a production context. To this end we introduce some notation. A vector of m inputs is denoted by $x = (x_1, \dots, x_m)$ and a vector of s outputs is denoted by $y = (y_1, \dots, y_s)$. A vector of m input prices

is denoted by $c = (c_1, \dots, c_m)$ and a vector of s output prices is denoted by $p = (p_1, \dots, p_s)$. The production technology is given by the set $T = \{(x, y) : x \in R_+^m, y \in R_+^s, x \text{ can produce } y\}$.

Obviously, not all input-output vectors belonging to the production technology are technologically efficient. Firms usually want to use the smallest levels of inputs to produce the greatest levels of outputs. To this respect, to measure efficiency it is necessary to compare the actual performance with a certain reference set of the production technology. We are actually referring to a subset of the boundary of T , defined as follows.

Definition 1. The weakly efficient subset of T , $\partial^W(T)$, is defined as

$$\partial^W(T) = \{(x, y) \in T : (-u, v) > (-x, y) \Rightarrow (u, v) \notin T\}^2.$$

We also assume that T is nonempty (P1), closed (P2), convex (P3) and satisfies strong disposability (P4). Postulate P2 guarantees that $\partial^W(T) \subset T$, P3 guarantees connectivity and continuity, and P4 guarantees that all $(x, y) \in \partial^W(T)$ satisfy a weak version of the Koopmans [5] efficiency condition.

In this paper, we think of the firm as a competitive profit maximizer. In other words, the firm takes prices as fixed and chooses a feasible production plan $(x, y) \in T$ which maximizes its profit. The resulting (optimum) profit is a function of the price vector (c, p) which we denote by $\Pi(c, p)$.

Definition 2. Given a vector of input and output prices $(c, p) \in R_+^m \times R_+^s$, and a production technology T , then the firm's profit function Π is defined as

² $(a, b) > (d, e)$ means that $a_i > d_i, \forall i = 1, \dots, m$ and $b_r > e_r, \forall r = 1, \dots, s$. In the same way, $(a, b) \geq (d, e)$ means that $a_i \geq d_i, \forall i = 1, \dots, m$ and $b_r \geq e_r, \forall r = 1, \dots, s$.

$$\Pi(c, p) = \sup_{x, y} \left\{ \sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i : (x, y) \in T \right\}.$$

Additionally, postulates P1-P4 are not very demanding and they allow to establish a duality between the profit function Π and the production technology T , with T defined by (see Färe and Primont [6])

$$T = \left\{ (x, y) \in R_+^m \times R_+^s : \sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \leq \Pi(c, p), \forall (c, p) \in R_+^m \times R_+^s \right\}.$$

Thanks to P2 and P3, and applying the separating hyperplane theorem, we know that for each $(x, y) \in \partial^W(T)$ there exists at least a shadow or intrinsic price vector $(c, p) \in R_+^m \times R_+^s$ such that $(u, v) \in T$ implies

$$\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \geq \sum_{r=1}^s p_r v_r - \sum_{i=1}^m c_i u_i. \text{ And, as a consequence of Definition 2,}$$

necessarily $\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i = \Pi(c, p)$. Therefore, the vector $(c, p, \Pi(c, p))$ defines a supporting hyperplane³ of T .

Now, we are ready to introduce the concept of a loss function.

Definition 3. Given $(x, y) \in \partial^W(T)$ and (c, p) , a shadow price vector of (x, y) , the loss function $L(x_0, y_0; x, y, c, p) : R_+^m \times R_+^s \rightarrow R$ is defined as

$$L(x_0, y_0; x, y, c, p) = \sum_{i=1}^m c_i (x_{i0} - x_i) + \sum_{r=1}^s p_r (y_r - y_{r0}),$$

where x_0 and y_0 are observed input and output vectors.

³ By definition, a supporting hyperplane of T contains at least one point of $\partial^W(T)$, and one of its two half-spaces contains all points of T . In general, given a generic point $(\tilde{x}, \tilde{y}) \in R_+^m \times R_+^s$, a vector $(c, p, \alpha) \in R_+^m \times R_+^s \times R$ defines a supporting hyperplane given by the next equation

$$\sum_{r=1}^s p_r \tilde{y}_r - \sum_{i=1}^m c_i \tilde{x}_i = \alpha.$$

In what follows, we are going to define a minimization model, which generalizes Debreu's problem, by considering a broader set of normalization conditions. Being more specific, we may consider a normalization condition which involves more than one restriction and, at the same time, each restriction may not involve necessarily all the shadow prices.

The loss function in Definition 3 becomes the objective function of the following loss function minimization model (A1):

$$\begin{aligned}
 L^*(x_0, y_0; NC(c, p)) = & \inf \sum_{i=1}^m c_i(x_{i0} - x_i) + \sum_{r=1}^s p_r(y_r - y_{r0}) \\
 \text{s.t.} & \\
 & \left. \begin{aligned}
 & (x, y) \in \partial^W(T) \\
 & (c, p) \text{ are shadow prices of } (x, y)
 \end{aligned} \right\} \\
 & \text{Normalization Condition } NC(c, p)
 \end{aligned}$$

We refer to $L^*(x_0, y_0; NC(c, p))$ as the optimal loss function corresponding to allocation (x_0, y_0) for the selected $NC(c, p)$. We assume that $NC(c, p)$ may be expressed through any number of restrictions. We further assume that the set of shadow prices that satisfy $NC(c, p)$ must be non-empty and closed and the zero-vector cannot belong to $NC(c, p)$, to make sure that the prices are not affected by an arbitrary positive scalar. Eventually, the nonlinear function $\sum_{i=1}^m c_i(-x_i) + \sum_{r=1}^s p_r y_r$ that appears as a part of the above objective function may also appear in the normalization condition $NC(c, p)$.

The objective function in the formulation of model A1 is a nonlinear function of an optimal allocation and its shadow prices, and, a priori, difficult to solve. Therefore, we want to develop an equivalent formulation that has a linear objective function.

First, observe that if (c, p) is a shadow price vector of (x, y) then

$$\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i = \Pi(c, p),$$

as we established above. Therefore, the following

model, A2, is equivalent to the loss function minimization model A1.

$$L^*(x_0, y_0; NC(c, p)) = \inf \Pi(c, p) - \left(\sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m c_i x_{i0} \right)$$

s.t.

$$\left\{ \begin{array}{l} (x, y) \in \partial^W(T) \\ (c, p) \text{ shadow prices of } (x, y) \end{array} \right\}$$

Normalization Condition $NC(c, p)$

Consequently, minimizing the difference between the profit function and the profit at the assessed point (x_0, y_0) , taking into account the mentioned constraints, yields the optimal loss function.

Secondly, we are able to obtain the optimal loss function $L^*(x_0, y_0; NC(c, p))$ by means of a model with a linear objective function, as Proposition 1 shows.

Proposition 1. Model A2 has the same optimal value as the next one, A3:

$$\inf - \sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha$$

s.t.

$$\left\{ \begin{array}{l} (c, p, \alpha) \in R_+^m \times R_+^s \times R \text{ defines} \\ \text{any supporting hyperplane of } T \end{array} \right\}$$

Normalization Condition $NC(c, p)$

Proof. Let (c^*, p^*, α^*) be an optimal solution of model A3. Then, since $(c^*, p^*, \alpha^*) \in R_+^m \times R_+^s \times R$ defines a supporting hyperplane of T , we have that there exists an $(x^*, y^*) \in \partial^W(T)$ such that $\sum_{r=1}^s p_r^* y_r^* - \sum_{i=1}^m c_i^* x_i^* = \alpha^*$ and $\sum_{r=1}^s p_r^* y_r^* - \sum_{i=1}^m c_i^* x_i^* \geq \sum_{r=1}^s p_r^* v_r - \sum_{i=1}^m c_i^* u_i, \forall (u, v) \in T$. Hence, by definition, (c^*, p^*) is a shadow price vector of (x^*, y^*) . Now, observe that $(x^*, y^*; c^*, p^*)$ is a feasible

solution of model A2. Let us suppose that there exists a feasible solution $(\bar{x}, \bar{y}; \bar{c}, \bar{p})$ of model A2 such that

$$\Pi(\bar{c}, \bar{p}) - \left(\sum_{r=1}^s \bar{p}_r y_{r0} - \sum_{i=1}^m \bar{c}_i x_{i0} \right) < \Pi(c^*, p^*) - \left(\sum_{r=1}^s p_r^* y_{r0} - \sum_{i=1}^m c_i^* x_{i0} \right).$$

Observe that the vector $(\bar{c}, \bar{p}, \Pi(\bar{c}, \bar{p}))$ defines a supporting hyperplane of T .

Then, defining $\bar{\alpha} = \Pi(\bar{c}, \bar{p})$ we have that $(\bar{c}, \bar{p}, \bar{\alpha})$ is a feasible solution of model

A3. And the objective value associated to this feasible solution is

$$\begin{aligned} -\sum_{r=1}^s \bar{p}_r y_{r0} + \sum_{i=1}^m \bar{c}_i x_{i0} + \bar{\alpha} &= \Pi(\bar{c}, \bar{p}) - \left(\sum_{r=1}^s \bar{p}_r y_{r0} - \sum_{i=1}^m \bar{c}_i x_{i0} \right) < \\ \Pi(c^*, p^*) - \left(\sum_{r=1}^s p_r^* y_{r0} - \sum_{i=1}^m c_i^* x_{i0} \right) &= -\sum_{r=1}^s p_r^* y_{r0} + \sum_{i=1}^m c_i^* x_{i0} + \alpha^*. \end{aligned}$$

The last implication contradicts that (c^*, p^*, α^*) is an optimal solution of model A3, which concludes the proof. ■

Consequently, from now on we will consider model A3 to calculate the optimal loss function, i.e.,

$$\begin{aligned} L^*(x_0, y_0; NC(c, p)) &= \inf -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\ &\text{s.t.} \\ &\left\{ \begin{array}{l} (c, p, \alpha) \in R_+^m \times R_+^s \times R \text{ defines} \\ \text{any supporting hyperplane of } T \end{array} \right\} \\ &\text{Normalization Condition } NC(c, p) \end{aligned}$$

Note also that in model A3 α is shadow profit $\Pi(c, p)$. Then, if it happens that

the nonlinear function $\sum_{i=1}^m c_i (-x_i) + \sum_{r=1}^s p_r y_r$ appears in $NC(c, p)$, we can replace

it by α , because $\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i = \Pi(c, p) = \alpha$, eliminating this nonlinear

expression from the normalization condition.⁴ Consequently, from now on, we will write $NC(c, p, \alpha)$ instead of $NC(c, p)$.

3. Deriving Families of Linear Efficiency Models

In what follows we restrict our analysis to either linear efficiency models or nonlinear efficiency models that can be linearized. In either case, in the linear loss function model we impose linearity on the normalization condition, i.e., it can be represented by means of a finite set of equalities and/or inequalities which are linear in c , p and α . In that case we write $LNC(c, p, \alpha)$ instead of $NC(c, p, \alpha)$. Moreover, from now on we assume that T is defined through a finite set of n homogeneous production units $\{(x_j, y_j), j = 1, \dots, n\}$, as any known DEA (Data Envelopment Analysis) efficiency model does. Therefore, T is defined precisely as

$$T = \left\{ (x, y) \in R_+^m \times R_+^s : (x, -y) \geq \sum_{j=1}^n \lambda_j (x_j, -y_j), \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\},$$

which allows the technology to satisfy variable returns to scale (VRS). If we want to restrict technology to satisfy non-increasing returns to scale (NIRS), non-decreasing returns to scale (NDRS) or constant returns to scale (CRS), we have to modify the constraint on the sum of the intensity variables λ_j in the definition of T as follows:

$$NIRS: \sum_{j=1}^n \lambda_j \leq 1; \quad NDRS: \sum_{j=1}^n \lambda_j \geq 1; \quad CRS: \sum_{j=1}^n \lambda_j \geq 0.$$

⁴ Model 7 of Section 3 provides an example of a minimized loss function model where the normalization condition has two constraints and where α appears in one of them instead of

$$\sum_{i=1}^m c_i (-x_i) + \sum_{r=1}^s p_r y_r.$$

Let us consider the next “linear loss function model”, A4, to evaluate the efficiency of a production unit (x_0, y_0) belonging to $\{(x_j, y_j), j = 1, \dots, n\}$.

$$\begin{aligned}
 & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 & \text{s.t.} \\
 & \quad \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & \quad c \geq 0_m, p \geq 0_s \\
 & \quad \text{Linear normalization condition LNC}(c, p, \alpha)
 \end{aligned}$$

We will show, by means of Proposition 2, that the optimal value of model A4 is exactly the optimal value of model A3 with a linear normalization condition.

Observe that we do not need to declare in model A4 that the set of hyperplanes we are considering are supporting hyperplanes, because the minimization process does the job for us, as Lemma 1 shows.

Lemma 1. Let (c^*, p^*, α^*) be an optimal solution of the linear loss function model A4. Then (c^*, p^*, α^*) defines a supporting hyperplane of T .

Proof. First, we prove that there exist $j' = 1, \dots, n$ such that

$\sum_{r=1}^s p_r^* y_{rj'} - \sum_{i=1}^m c_i^* x_{ij'} - \alpha^* = 0$. Otherwise, we would get a strict inequality in each of

the first n constraints, $\sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0$, and defining β^* as

$$\beta^* = \min \left\{ -\left(\sum_{r=1}^s p_r^* y_{rj} - \sum_{i=1}^m c_i^* x_{ij} - \alpha^* \right) > 0, j = 1, \dots, n \right\},$$

the new feasible solution $(c^*, p^*, \alpha^* - \beta^*)$ would produce an objective function value lower than that achieved by the initial optimal solution (c^*, p^*, α^*) , which is

a contradiction. Consequently, the hyperplane associated with the vector (c^*, p^*, α^*) contains the point $(x_j, y_j) \in T$. Moreover, since $\sum_{r=1}^s p_r^* y_{rj} - \sum_{i=1}^m c_i^* x_{ij} - \alpha^* \leq 0$, $j=1, \dots, n$, the n points that define the technology belong to the same halfspace. To end the proof, it is easy to show that any point of T belongs also to the same halfspace, i.e., the hyperplane associated with (c^*, p^*, α^*) is a supporting hyperplane of T . ■

Finally, Proposition 2 shows that model A4 collapses to model A3 provided that the normalization condition is linear.

Proposition 2. The linear loss function model A4 has the same optimal value and optimal solutions α as the loss function model A3 when we assume linear normalization conditions.

Proof. First, let us prove that a feasible (c, p, α) of A3 with $LNC(c, p, \alpha)$ is also feasible for A4. Knowing that $\alpha = \Pi(c, p)$ we get that $\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \leq \alpha$, $\forall (x, y) \in T$. In particular, $\{(x_j, y_j), j=1, \dots, n\} \subset T$ satisfy the last inequality, i.e., (c, p, α) is feasible for A4. Hence, since both models have the same objective function, the optimal value of A3 is greater than or equal to the optimal value of A4. Secondly, let us prove that both models have the same optimal value. By Lemma 1, the optimal solution of A4 is a feasible solution of A3, which implies that the last obtained inequality becomes an equality, as desired. ■

Since in the linear loss function model, A4, α is free, the supporting hyperplane has an intercept unrestricted in sign. This corresponds to the VRS technology specified above, and we restrict our subsequent analysis to VRS models. Nevertheless, the generation of models under other returns to scale assumptions is straightforward. For a NIRS model we add to the above model the condition $\alpha \geq 0$; for a NDRS model we add the condition $\alpha \leq 0$; and for a

CRS model we add the condition $\alpha = 0$, or, equivalently, we delete α everywhere.

Notes

- 1) The BCC model of Banker et al. [7] is known as the first VRS-DEA model. Nonetheless, VRS models appeared in the economics literature on efficiency measurement as early as 1972, as ten Raa [8] points out. In the DEA framework, the NIRS and NDRS models were introduced by Färe et al. [9], and the first CRS model is the CCR model of Charnes et al. [10].
- 2) In all considered linear loss function models $x \in R_+^m \setminus \{0_m\}$, $y \in R_+^s \setminus \{0_s\}$. However in the objective function it is possible to conduct the minimization over fewer than m inputs and/or fewer than s outputs. Such a framework corresponds to a money metric measure of subvector efficiency, or of efficiency in the presence of non-discretionary or quasi-fixed variables. We do not highlight this possibility, but we remind the reader of its existence.

3.1. Radial DEA Models and Directional Distance Function Models

Radial DEA models have evolved from Debreu's [1] coefficient of resource utilization and Farrell's [2] measure of technical efficiency, and involve scaling observed quantity vectors. Directional distance function models have evolved from Debreu's loss function and Luenberger's [11, 12] benefit and shortage functions, and involve translating observed quantity vectors.

2008 constituted the 30th birthday of DEA. This paper presents a new unifying way of dealing with any DEA model, exploring the structure of the multiplier form, also referred to as the dual (linear) program. As we mentioned above, the only difference between any pair of DEA models is the finite set of restrictions grouped as "normalization conditions".

Model 1. The BCC input-oriented model

We next show the linear duals of the BCC input-oriented model.

Envelopment form

$$\begin{aligned}
 & \text{Min} \quad \theta \\
 & \text{s.t.} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{i0}, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \text{Max} \quad \sum_{r=1}^s p_r y_{r0} - \alpha \\
 & \text{s.t.} \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & \sum_{i=1}^m c_i x_{i0} = 1 \\
 & c_i \geq 0, \quad i = 1, \dots, m \\
 & p_r \geq 0, \quad r = 1, \dots, s
 \end{aligned}$$

In this case, a comparison of the multiplier form and the linear loss function model suggests that the corresponding linear normalization condition is

$$\sum_{i=1}^m c_i x_{i0} = 1 \quad (LNC1).$$

In this case, the normalization condition only involves input prices. It is easy to show (see Appendix) that by manipulating the corresponding linear loss function

model, we have at optimum $1 - L^*(x_0, y_0; LNC1) = \sum_{r=1}^s p_r^* y_{r0} - \alpha^*$, which specifies

that 1 minus the optimal linear loss function equals the optimal objective function of the BCC input-oriented model. Since the minimized loss function measures inefficiency, the maximized objective of the BCC model measures efficiency. In particular, if the rated unit is efficient, we get

$$\sum_{r=1}^s p_r^* y_{r0} - \alpha^* = 1 \Leftrightarrow L^*(x_0, y_0; LNC1) = 0.$$

Related models

- 1) The CCR input-oriented model.
- 2) The radial NIRS and NDRS input-oriented models.

- 3) The (nonlinear) CRS hyperbolic model of Färe et al. [9], because it can be linearized to the CCR input-oriented model.
- 4) The AR (Assurance Region) input-oriented models, introduced by Thompson et al. [13], are particular cases of CCR models in the next sense. The multiplier form of an AR model include all the restrictions of the CCR model, and, particularly, the linear normalization condition, together with a set of “value judgment restrictions” such as $\left\{ \frac{c_i}{c_1} \geq k_i, i = 2, \dots, m \right\}$, where $k_i > 0, i = 1, \dots, m$. Our linear loss function model can be easily extended to cover the AR models, just by adding the same value judgment restrictions.

Model 2. The BCC output-oriented model

We next show the envelopment form and multiplier form of the BCC output-oriented model.

Envelopment form

$$\begin{aligned}
 & \text{Max} \quad \phi \\
 & \text{s.t.} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \phi y_{r0}, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \text{Min} \quad \sum_{i=1}^m c_i x_{i0} + \alpha \\
 & \text{s.t.} \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & \sum_{r=1}^s p_r y_{r0} = 1 \\
 & c_i \geq 0, \quad i = 1, \dots, m \\
 & p_r \geq 0, \quad r = 1, \dots, s
 \end{aligned}$$

The corresponding linear normalization condition normalizes p only

$$\sum_{r=1}^s p_r y_{r0} = 1 \quad (\text{LNC2}),$$

and by manipulating the linear loss function model with *LNC2*, we have at optimum $1 + L^*(x_0, y_0; LNC2) = \sum_{i=1}^m c_i^* x_{i0} + \alpha^*$, which specifies the relationship between the optimal loss function and the optimal objective function of the BCC output-oriented model (see the Appendix). For any efficient point, $\sum_{i=1}^m c_i^* x_{i0} + \alpha^* = 1 \Leftrightarrow L^*(x_0, y_0; LNC2) = 0$.

Related models

- 5) The CCR output-oriented model
- 6) The radial NIRS and NDRS output-oriented models
- 7) The (nonlinear) CRS hyperbolic model, because it can be linearized to the CCR output-oriented model
- 8) The AR output-oriented models are extensions of the CCR output-oriented model, just with the same reasoning as for the input-oriented case.

Model 3. The directional distance function model

We show the envelopment form of the directional distance function model and the corresponding never used DEA multiplier form, given $g = (g^-, g^+) \in R_+^m \times R_+^s$ a nonzero pre-specified vector called the directional vector.

Envelopment form

$$\begin{aligned}
 & \text{Max} \quad \beta \\
 & \text{s.t.} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} - \beta g_i^-, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} + \beta g_r^+, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 & \text{s.t.} \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & \sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c_i g_i^- = 1 \\
 & c_i \geq 0, \quad i = 1, \dots, m \\
 & p_r \geq 0, \quad r = 1, \dots, s
 \end{aligned}$$

In this case, the corresponding linear normalization condition normalizes both p and c as follows

$$\sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c_i g_i^- = 1 \quad (LNC3),$$

We want to point out that g may depend on $\{(x_j, y_j), j = 1, \dots, n\}$.

Exploiting the dual of the linear loss function model with $LNC3$, we have at optimum $L^*(x_0, y_0; LNC3) = \beta^*$, which describes the relationship between the optimal loss function and the shortage function of Luenberger [11], and which Chambers et al. [14] refer to as the directional technology distance function⁵. Observe that both functions provide a measure of inefficiency, and so $\beta^* = 0 \Leftrightarrow L^*(x_0, y_0; LNC3) = 0$.

Related models

⁵ Chambers et al. [14] proved that there is a dual relationship between the profit function and the directional distance function in a production context. In particular, the directional distance function can be obtained from the profit function as follows:

$$\beta^* = \inf_{(c, p) \geq (0_m, 0_s)} \left\{ \Pi(c, p) - \left(\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \right) : \sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c_i g_i^- = 1 \right\}.$$

Our model A2 resembles this last dual relation but instead of considering any nonnegative price vector considers only the shadow price vectors associated with points belonging to the weakly efficient subset of T .

- 9) The CRS directional distance function model.
- 10) The NIRS and NDRS directional distance function models.
- 11) The ℓ_∞ distance to $\partial^W(T)$ (Briec [15]), setting $g = (1_m, 1_s)$. As a reminder,

$$D_{\ell_\infty}((\bar{x}, \bar{y}), (x_0, y_0)) = \max\{|\bar{x}_i - x_{i0}|, i = 1, \dots, m, |\bar{y}_r - y_{r0}|, r = 1, \dots, s\}.$$
- 12) The Range Directional Model (RDM) of Silva Portela et al. [16], which considers an ideal point associated with the set of n units, defined as $z_i^- = \min_j x_{ji}, i = 1, \dots, m, z_r^+ = \max_j y_{jr}, r = 1, \dots, s$, so as to define the data-dependent directional vector $g_i^- = x_{i0} - z_i^-, i = 1, \dots, m, g_r^+ = z_r^+ - y_{r0}, r = 1, \dots, s$.
- 13) Observe that taking $g = (g^-, 0_s)$ or $g = (0_m, g^+)$ generate two special cases of Luenberger's shortage function, which Chambers et al. [14, 17] refer to as directional input and output distance functions, respectively. Observe also that models 1 and 2 can indirectly be obtained as particular cases, by setting $g = (x_0, 0_s)$, to generate the inefficiency associated with the BCC input-oriented model, and choosing $g = (0_m, y_0)$ to perform the same task for the BCC output-oriented model.
- 14) Briec [18] introduced by means of a linear program "a Graph-type extension of Farrell technical efficiency measure", which at the end is exactly a specific directional distance function model, obtained by considering as directional vector $g = (x_0, y_0)$. Consequently, the corresponding normalization condition is $\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} = 1$.

3.2. Additive DEA Models

We consider the weighted additive model of Lovell and Pastor [19], which has the same restrictions as the additive model of Charnes et al. [20], but its

objective function is modified through the assignment of weights $(w^-, w^+) \in R_+^m \times R_+^s$ to the input slacks and the output slacks. The weights can vary across production units. This allows us to generate a wide range of additive models in a unified way.

Model 4. The weighted additive model

The envelopment form and the multiplier form of the weighted additive model are, respectively, as follows.

Envelopment form

$$\begin{aligned}
 & \text{Max} \quad \sum_{i=1}^m w_i^- s_{i0}^- + \sum_{r=1}^s w_r^+ s_{r0}^+ \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad i = 1, \dots, m \\
 & \quad \quad \sum_{j=1}^n \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad r = 1, \dots, s \\
 & \quad \quad \sum_{j=1}^n \lambda_j = 1 \\
 & \quad \quad \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & \quad \quad s_{i0}^- \geq 0, \quad i = 1, \dots, m \\
 & \quad \quad s_{r0}^+ \geq 0, \quad r = 1, \dots, s
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & \quad \quad c_i \geq w_i^-, \quad i = 1, \dots, m \\
 & \quad \quad p_r \geq w_r^+, \quad r = 1, \dots, s
 \end{aligned}$$

The corresponding linear normalization condition is

$$\{c \geq w^-, p \geq w^+\} \quad (LNC4),$$

and by manipulating the linear loss function model with *LNC4*, we have at

optimum $L^*(x_0, y_0; LNC4) = \sum_{i=1}^m w_i^- s_{i0}^{*-} + \sum_{r=1}^s w_r^+ s_{r0}^{*+}$, which specifies the relationship

between the optimal loss function and the optimal objective function of the weighted additive model (see the Appendix). Observe that in this case the normalization condition is a set of linear inequalities. The maximized objective

function of the weighted additive model measures inefficiency, just as the minimized loss function does. Hence, for any efficient unit, $L^* = 0 \Leftrightarrow s_{i_0}^- = s_{r_0}^+ = 0, \forall i = 1, \dots, m, \forall r = 1, \dots, s$.

Related models

- 15) The CRS weighted additive model of Ali and Seiford [21]
- 16) The NIRS and NDRS weighted additive models
- 17) The (standard) additive model of Charnes et al. [20], which takes all the weights equal to 1
- 18) The enhanced additive model of Charnes et al. [22], also called MIP (Measure of Inefficiency Proportions) in Cooper et al. [23],⁶ which takes $w_i^- = \frac{1}{x_{i_0}}, i = 1, \dots, m, w_r^+ = \frac{1}{y_{r_0}}, r = 1, \dots, s$ and requires all quantities to be strictly positive.
- 19) The normalized weighted additive model of Lovell and Pastor [19], which takes $w_i^- = \frac{1}{\sigma_i^-}, i = 1, \dots, m, w_r^+ = \frac{1}{\sigma_r^+}, r = 1, \dots, s$, where σ_i^- and σ_r^+ are the standard deviations of inputs and outputs over the n production units.
- 20) The RAM (Range Adjusted Measure) of inefficiency model of Cooper et al. [23], takes $w_i^- = \frac{1}{(m+s)R_i^-}, i = 1, \dots, m, w_r^+ = \frac{1}{(m+s)R_r^+}, r = 1, \dots, s$, where R_i^- and R_r^+ are the ranges of inputs and outputs over the n production units. In contrast with any weighted additive model, the RAM model is designed to

⁶ Based on the solution of the enhanced additive model, Bardhan et al. [24] defined an efficiency measure called MED (measure of efficiency dominance) which was renamed by Banker and Cooper [25] as MEP (measure of efficiency proportions), i.e., MEP=MED.

define easily an efficiency measure. In fact, Cooper et al. [23, p. 20] define the corresponding measure of efficiency as 1-RAM.

3.3. Russell Models

Russell models were introduced, and named, by Färe and Lovell [26] as a way of projecting, in a non-radial way, an observed allocation to the strongly efficient subset of technology.

Model 5. The input-oriented Russell measure

Next we show the envelopment form of the input-oriented Russell measure and the corresponding never used dual linear form.

Envelopment form

$$\begin{aligned}
 & \text{Min} \quad \frac{1}{m} \sum_{i=1}^m \theta_i \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{i0}, \quad i = 1, \dots, m \\
 & \quad \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad r = 1, \dots, s \\
 & \quad \sum_{j=1}^n \lambda_j = 1 \\
 & \quad \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & \quad \theta_i \in R, \quad i = 1, \dots, m
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \text{Max} \quad \sum_{r=1}^s p_r y_{r0} - \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & \quad c_i x_{i0} = \frac{1}{m}, \quad i = 1, \dots, m \\
 & \quad c_i \geq 0, \quad i = 1, \dots, m \\
 & \quad p_r \geq 0, \quad r = 1, \dots, s
 \end{aligned}$$

In this case, the corresponding linear normalization condition is

$$c_i x_{i0} = \frac{1}{m}, \quad i = 1, \dots, m \quad (\text{LNC5}).$$

Considering the dual of the linear loss function model with *LNC5*, changing the orientation of the objective function and adding to it 1, and performing a change of variables (see the Appendix), we have at optimum

$$1 - L^*(x_0, y_0; LNC5) = \frac{1}{m} \sum_{i=1}^m \theta_i^*,$$

which specifies the relationship between the optimal loss function and the optimal objective function of the VRS input-oriented Russell measure. As with Model 1, the objective function of this model measures efficiency, in contrast to the loss function. In this case, $L^* = 0 \Leftrightarrow \theta_i^* = 1, \forall i = 1, \dots, m$.

Related models

21) The NIRS, NDRS and CRS input-oriented Russell measures.

Model 6. The output-oriented Russell measure

Next we show the envelopment form of the output-oriented Russell measure and the corresponding never used multiplier form.

Envelopment form

$$\begin{aligned} \text{Max} \quad & \frac{1}{s} \sum_{r=1}^s \phi_r \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq \phi_r y_{r0}, \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \\ & \phi_r \in \mathbb{R}, \quad r = 1, \dots, s \end{aligned}$$

Multiplier form

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^m c_i x_{i0} + \alpha \\ \text{s.t.} \quad & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\ & p_r y_{r0} = \frac{1}{s}, \quad r = 1, \dots, s \\ & c_i \geq 0, \quad i = 1, \dots, m \\ & p_r \geq 0, \quad r = 1, \dots, s \end{aligned}$$

The corresponding linear normalization condition is

$$p_r y_{r0} = \frac{1}{s}, \quad r = 1, \dots, s \quad (LNC6) .$$

Considering the dual of the linear loss function model with *LNC6*, and performing similar transformations as in Model 5 (see the Appendix), we have at optimum

$$1 + L^*(x_0, y_0; LNC6) = \frac{1}{s} \sum_{r=1}^s \phi_r^* ,$$

which specifies the relationship between the optimal loss function and the optimal objective function of the VRS output-oriented Russell measure. As in Model 5, the objective function measures efficiency, in contrast to the loss function. Now, $L^* = 0 \Leftrightarrow \phi_r^* = 1, \forall r = 1, \dots, s$.

Related models

22) The NIRS, NDRS and CRS output-oriented Russell measures.

Model 7. The enhanced Russell graph measure

Next we show the envelopment form of the VRS enhanced Russell graph measure as was defined in Pastor et al. [27].

$$\begin{aligned}
\text{Min} \quad & \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{r=1}^s \phi_r} \\
\text{s.t.} \quad & \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{i0}, \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \phi_r y_{r0}, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \theta_i \leq 1, \quad i = 1, \dots, m \\
& \phi_r \geq 1, \quad r = 1, \dots, s
\end{aligned}$$

Observe that the above measure is a nonlinear model. Nonetheless, it can be linearized by means of a change of variables (see Pastor et al. [27]).

In this case, the corresponding linear normalization condition is

$$\left\{ \begin{array}{l} c_i x_{i0} \geq \frac{1}{m}, i = 1, \dots, m \\ p_r y_{r0} \geq \frac{1}{s} \left(1 + \sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m c_i x_{i0} - \alpha \right), r = 1, \dots, s \end{array} \right\} \quad (LNC7) .$$

Considering the dual of the linear loss function model with *LNC7*, and performing several transformations, we have at optimum (see the Appendix)

$$1 - L^*(x_0, y_0; LNC7) = \frac{\frac{1}{m} \sum_{i=1}^m \theta_i^*}{\frac{1}{s} \sum_{r=1}^s \phi_r^*},$$

which specifies the relationship between the optimal loss function and the objective function of the fractional form of the VRS enhanced Russell graph measure of Pastor et al. [27]. As in Models 5 and 6, the optimal objective

function measures efficiency, in contrast to the optimal linear loss function. Here we get $L^* = 0 \Leftrightarrow \theta_i^* = \phi_r^* = 1, \forall i = 1, \dots, m, \forall r = 1, \dots, s$.

Related models

23) The NIRS, NDRS and CRS enhanced Russell graph measures.

4. Achieving units invariance through $LNC(c, p, \alpha)$

Units invariance is a highly desirable property for any efficiency measure. As a matter of fact, resorting to a DEA efficiency model that is not units-invariant only allows one to categorize units as being (weakly) efficient or inefficient, but does not allow one to consider the efficiency scores as efficiency measures because they are not well defined. If we rescale inputs and/or outputs, efficiency scores and rankings of inefficient producers are affected. It is therefore important to verify under which conditions the final linear loss function model, A4, of section 3 is units invariant.

Proposition 3. The linear loss function model, apart from $LNC(c, p, \alpha)$, is units invariant.

Proof. Considering the linear loss function model of section 3, A4, we observe that the set of coefficients of the supporting hyperplanes $C := \{(c, p, \alpha) \in R_+^m \times R_+^s \times R\}$ can also be represented as

$$C = \left\{ (\hat{c}, \hat{p}, \hat{\alpha}) \in R^{m+s+1} : \hat{c}_i = \gamma_i c_i, \hat{p}_r = \gamma_{m+r} p_r, \right. \\ \left. \gamma_i > 0, i = 1, \dots, m, \gamma_{m+r} > 0, r = 1, \dots, s, \hat{\alpha} = \alpha, \forall (c, p) \geq 0_{m+s} \right\}.$$

(Observe that there are as many representations as vectors of γ 's.) Then it is apparent that the objective function and the specified restrictions of the linear loss function model, apart from the normalization condition, are units invariant, since the set of specific positive multiplicative factors that scale each of the

inputs ($\gamma_i > 0, i = 1, \dots, m$) and outputs ($\gamma_{m+r} > 0, r = 1, \dots, s$) when their units are changed can be absorbed by the shadow prices. Additionally, note that, in the part of the model we are considering, α remains unchanged so we take $\hat{\alpha} = \alpha$.

■

Observe that in the objective function and in the specified restrictions of the linear loss function model, the coefficients of the shadow prices are homogeneous functions of degree +1 in $\{x_{ij}, i = 1, \dots, m, y_{rj}, r = 1, \dots, s, j = 1, \dots, n\}$, and that the coefficient of α is homogeneous of degree 0. Proposition 4 shows that the same requirements applied to the $LNC(c, p, \alpha)$ leads to a units-invariant model. In order to study more precisely $LNC(c, p, \alpha)$, let us consider, without loss of generality, that the linear normalization condition is given by means of the inequality

$$\begin{aligned} & \sum_{i=1}^m f_i((x_{11}, \dots, x_{m1}, y_{11}, \dots, y_{s1}), \dots, (x_{1n}, \dots, x_{mn}, y_{1n}, \dots, y_{sn}))c_i + \\ & \sum_{r=1}^s f_{m+r}((x_{11}, \dots, x_{m1}, y_{11}, \dots, y_{s1}), \dots, (x_{1n}, \dots, x_{mn}, y_{1n}, \dots, y_{sn}))p_r + \\ & + f_{m+r+1}((x_{11}, \dots, x_{m1}, y_{11}, \dots, y_{s1}), \dots, (x_{1n}, \dots, x_{mn}, y_{1n}, \dots, y_{sn}))\alpha \geq 1. \end{aligned}$$

This expression can be reformulated by reordering the arguments of any f_p so as to get subvectors associated with each input and each output of the model. Hence let us define

$$\begin{aligned} & g_k((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn})) = \\ & f_k((x_{11}, \dots, x_{m1}, y_{11}, \dots, y_{s1}), \dots, (x_{1n}, \dots, x_{mn}, y_{1n}, \dots, y_{sn})), k = 1, \dots, m + s + 1 \end{aligned}$$

Proposition 4. Let us consider that $LNC(c, p, \alpha)$ is given as

$$\begin{aligned} & \sum_{i=1}^m g_i \left((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn}) \right) c_i + \\ & \sum_{r=1}^s g_{m+r} \left((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn}) \right) p_r + \\ & + g_{m+r+1} \left((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn}) \right) \alpha \geq 1. \end{aligned}$$

The next condition is sufficient for $LNC(c, p, \alpha)$ to be units invariant.

g_i is homogeneous of degree +1 in (x_{i1}, \dots, x_{in}) and of degree 0 in the rest of the $m+s-1$ subvectors, for $i=1, \dots, m$, g_{m+r} is homogeneous of degree +1 in (y_{r1}, \dots, y_{rn}) and of degree 0 in the rest of the $m+s-1$ subvectors, for $r=1, \dots, s$, and g_{m+r+1} is of degree 0 in all the $m+s$ subvectors.

Proof. Let us consider the next variable change (of inputs and outputs):

$$\hat{x}_{ij} = \gamma_i \cdot x_{ij}, \gamma_i > 0, i = 1, \dots, m, \hat{y}_{rj} = \gamma_{m+r} \cdot y_{rj}, \gamma_{m+r} > 0, r = 1, \dots, s, j = 1, \dots, n.$$

The new expression for the linear normalization condition is

$$\begin{aligned} & \sum_{i=1}^m g_i \left((\hat{x}_{11}, \dots, \hat{x}_{1n}), \dots, (\hat{x}_{m1}, \dots, \hat{x}_{mn}), (\hat{y}_{11}, \dots, \hat{y}_{1n}), \dots, (\hat{y}_{s1}, \dots, \hat{y}_{sn}) \right) c_i + \\ & \sum_{r=1}^s g_{m+r} \left((\hat{x}_{11}, \dots, \hat{x}_{1n}), \dots, (\hat{x}_{m1}, \dots, \hat{x}_{mn}), (\hat{y}_{11}, \dots, \hat{y}_{1n}), \dots, (\hat{y}_{s1}, \dots, \hat{y}_{sn}) \right) p_r + \\ & + g_{m+r+1} \left((\hat{x}_{11}, \dots, \hat{x}_{1n}), \dots, (\hat{x}_{m1}, \dots, \hat{x}_{mn}), (\hat{y}_{11}, \dots, \hat{y}_{1n}), \dots, (\hat{y}_{s1}, \dots, \hat{y}_{sn}) \right) \alpha \geq 1. \end{aligned}$$

According to the conditions specified in Proposition 4 we get

$$\begin{aligned} & g_k \left((\hat{x}_{11}, \dots, \hat{x}_{1n}), \dots, (\hat{x}_{m1}, \dots, \hat{x}_{mn}), (\hat{y}_{11}, \dots, \hat{y}_{1n}), \dots, (\hat{y}_{s1}, \dots, \hat{y}_{sn}) \right) = \\ & \gamma_k g_k \left((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn}) \right), k = 1, \dots, m + s, \end{aligned}$$

and

$$\begin{aligned} & g_{m+r+1} \left((\hat{x}_{11}, \dots, \hat{x}_{1n}), \dots, (\hat{x}_{m1}, \dots, \hat{x}_{mn}), (\hat{y}_{11}, \dots, \hat{y}_{1n}), \dots, (\hat{y}_{s1}, \dots, \hat{y}_{sn}) \right) = \\ & g_{m+r+1} \left((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn}) \right). \end{aligned}$$

Finally, and due to the relation between $(\hat{c}, \hat{p}, \hat{\alpha})$ and (c, p, α) established in the proof of Proposition 3, we get that $LNC(\hat{c}, \hat{p}, \hat{\alpha})$ as a function of $((\hat{x}_{11}, \dots, \hat{x}_{1n}), \dots, (\hat{x}_{m1}, \dots, \hat{x}_{mn}), (\hat{y}_{11}, \dots, \hat{y}_{1n}), \dots, (\hat{y}_{s1}, \dots, \hat{y}_{sn}))$ equals $LNC(c, p, \alpha)$ as a function of $((x_{11}, \dots, x_{1n}), \dots, (x_{m1}, \dots, x_{mn}), (y_{11}, \dots, y_{1n}), \dots, (y_{s1}, \dots, y_{sn}))$, which means that the two loss function models before and after the change of variables are exactly the same, i.e., the loss function model is units invariant. ■

Notes

1) Observe that the constant function $g(z) = k$, $k \neq 0$, is homogeneous of degree 0, while the constant function $g(z) = 0$ is homogeneous of any degree and, in particular, of degrees 0 and +1.

2) In particular, Proposition 4 holds if each shadow price appears multiplied by its corresponding input or output quantity, or by 0, and α appears multiplied by a constant which can be 0. The last condition is quite common, as we will see in what follows.

Now we are prepared to review the seven generated models and conclude if they are units invariant or not.

$$\underline{\text{Model 1:}} \quad \sum_{i=1}^m c_i x_{i0} = 1 \quad (\text{LNC1})$$

It is units invariant because Note 2 applies.

$$\underline{\text{Model 2:}} \quad \sum_{r=1}^s p_r y_{r0} = 1 \quad (\text{LNC2})$$

It is units invariant, with the same argument as for Model 1.

Model 3:
$$\sum_{r=1}^s p_r g_r^- + \sum_{i=1}^m c g_i^+ = 1 \quad (LNC3)$$

It is not always units invariant. If the components of the directional vector are related to the observations $\{(x_j, y_j) : j = 1, \dots, n\}$, as specified in Proposition 4, it is obvious that the directional distance function model achieves units invariance. This is the case of the RDM model of Silva Portela et al. [16]. On the other hand, if the components of the directional vector are non-zero constants, they do not satisfy Proposition 4 because they are functions homogeneous of degree 0. This is the case of the directional distance function model of Chambers et al. [14, 17]. Moreover, it is not difficult to work out a counterexample.

Counterexample.

Let us consider the simplest one input – one output case, under CRS, with the two units (1,1) and (4,2) and the directional vector $g = (1,1)$. On the x-y plane it is easy to see that the frontier corresponds to the semi-ray $y = x, x \geq 0$, which contains the efficient point (1,1) and that the projection of our second point on the frontier is the point (3,3), with $\beta^* = 1$. Now we change the x-scale so that point (1,1) becomes (1/2,1). Then the new frontier is given by $y = 2x$ and the projection of (2,2), the second rescaled point, is now (4/3,8/3) which corresponds to $\beta^* = 2/3$ for the same directional vector. The value of β^* has changed, i.e., the model is not units invariant.

Model 4:
$$c \geq w^-, p \geq w^+ \quad (LNC4)$$

It is not always units invariant for the same reason as Model 3, considering the vector of weights instead of the directional vector (see Lovell and Pastor [19]). If the weights are properly related to the observations $\{(x_j, y_j) : j = 1, \dots, n\}$, as happens with the enhanced additive, the RAM or the Lovell-Pastor models, then the weighted additive model achieves units invariance. These three related

models satisfy Proposition 4, as can be easily seen by rewriting *LNC4* as

$$c_i \frac{1}{w_i^-} \geq 1, i = 1, \dots, m, p_r \frac{1}{w_r^+} \geq 1, r = 1, \dots, s.$$

Model 5:
$$c_i x_{i0} = \frac{1}{m}, i = 1, \dots, m \quad (\text{LNC5})$$

It is units invariant by virtue of satisfying the condition of Note 2.

Model 6:
$$p_r y_{r0} = \frac{1}{s}, r = 1, \dots, s \quad (\text{LNC6})$$

It is units invariant by virtue of satisfying the condition of Note 2.

Model 7:

$$\left. \begin{array}{l} c_i x_{i0} \geq \frac{1}{m} \\ p_r y_{r0} \geq \frac{1}{s} \left(1 + \sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m c_i x_{i0} - \alpha \right) \end{array} \right\} \quad (\text{LNC7})$$

It is units invariant because if we rewrite the two normalization restrictions by transposing the denominators, we get two expressions that satisfy the condition of Proposition 4.

5. Conclusions

Debreu's coefficient of resource utilization has attracted considerable attention through the years, but his dual loss function has been largely neglected. This oversight is unfortunate, and in this paper we demonstrate just one analytical use to which the loss function can be put. We narrow our focus from an economy to its production activity, in which case the loss function provides a money metric measure of the value sacrificed to production inefficiency.

A generic loss function model, A1, appears in section 2. It follows directly from Debreu's formulation, and contains a linear objective function, a feasibility condition, and a not-necessarily linear normalization condition. This model is completely linearized early in section 3, for computational and theoretical considerations, and is the model, A4, we work with. It contains a linear objective function, a set of linear inequalities describing feasibility conditions, and a linear normalization condition.

In section 3 we use this linear loss function model to generate several popular families of linear efficiency models. A structural feature of these models is that all of them share the same linear objective function and the same set of linear inequalities describing feasibility conditions. What varies across models is the structure of the linear normalization condition. Simply by varying this condition in predetermined ways we are able to derive all known DEA families of linear efficiency models. In section 4 we show that some of these families of models share the critical property of units invariance, and we also show that some families of models are units-invariant only under restrictive conditions, while one family of models is not units-invariant.

An important implication of our analysis is that the derivation of linear efficiency models need not be an *ad hoc* exercise. By resurrecting Debreu's loss function we have provided an analytical framework within which any, currently known or still unknown, linear efficiency model can be derived.

APPENDIX

3.1. Radial DEA and Directional Distance Function Models

Model 1. The BCC input-oriented model

Let us consider the linear loss function model with our first linear normalization condition, $LNC1$.

$$\begin{aligned}
 L^*(x_0, y_0; LNC1) = & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 \text{s.t.} & \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & c \geq 0_m, p \geq 0_s \\
 & \sum_{i=1}^m c_i x_{i0} = 1 \quad (LNC1)
 \end{aligned}$$

Observe that, as a consequence of $LNC1$ the objective function of model 1 is equivalent to $1 + \text{Min} \left\{ -\sum_{r=1}^s p_r y_{r0} + \alpha \right\}$ or to $1 - \text{Max} \left\{ \sum_{r=1}^s p_r y_{r0} - \alpha \right\}$. Hence, we attain the following result.

$$\begin{aligned}
 1 - L^*(x_0, y_0; NC1) = & \text{Max} \quad \sum_{r=1}^s p_r y_{r0} - \alpha \\
 \text{s.t.} & \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & c \geq 0_m, p \geq 0_s \\
 & \sum_{i=1}^m c_i x_{i0} = 1 \quad (LNC1)
 \end{aligned}$$

This model is exactly the multiplier form of the Banker *et al.* [7] BCC input-oriented model. Being linear duals, the optimal value of the envelopment form, which is a finite number, matches with the optimal value of the multiplier form. Therefore, the next relation holds: $1 - L^*(x_0, y_0; LNC1) = \theta^*$.

Model 2. The BCC output-oriented model

Let us consider the linear loss function model with our second linear normalization condition, $LNC2$.

$$\begin{aligned}
 L^*(x_0, y_0; LNC2) = & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 \text{s.t.} & \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & c \geq 0_m, p \geq 0_s \\
 & \sum_{r=1}^s p_r y_{r0} = 1 \quad (LNC2)
 \end{aligned}$$

Observe that, as a consequence of $LNC2$, this second model is equivalent to the following one:

$$\begin{aligned}
 1 + L^*(x_0, y_0; LNC2) = & \text{Min} \quad \sum_{i=1}^m c_i x_{i0} + \alpha \\
 \text{s.t.} & \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & c \geq 0_m, p \geq 0_s \\
 & \sum_{r=1}^s p_r y_{r0} = 1 \quad (LNC2)
 \end{aligned}$$

This model is exactly the multiplier form of the output-oriented BCC model. Being linear duals, the optimal value of the envelopment form matches with the optimal value of the multiplier form. Therefore, the next relation holds:

$$1 + L^*(x_0, y_0; LNC2) = \phi^* .$$

Model 3. The directional distance function model

Let us consider the linear loss function model with our third linear normalization condition, *LNC3*.

$$\begin{aligned} L^*(x_0, y_0; LNC3) = & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\ \text{s.t.} \quad & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\ & c \geq 0_m, p \geq 0_s \\ & \sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c g_i^- = 1 \quad (LNC3) \end{aligned}$$

where $g = (g^-, g^+) \in R_+^m \times R_+^s$.

This model coincides with the multiplier form of the VRS directional distance function model (see Section 3, Model 3) due to Chambers *et al.* [14, 17]. Observe then that $L^*(x_0, y_0; LNC3) = \beta^*$, i.e., it provides a measure of technical inefficiency.

3.2. Additive DEA Models

Model 4. The weighted additive model

Let us consider the linear loss function model with our fourth linear normalization condition, *LNC4*.

$$\begin{aligned}
L^*(x_0, y_0; LNC4) = & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
& \text{s.t.} \\
& \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
& c \geq 0_m, p \geq 0_s \\
& c \geq w^-, p \geq w^+ \quad (LNC4)
\end{aligned}$$

Obviously, in this case the non-negativity conditions on the shadow prices are superfluous. We obtain directly the multiplier form of the weighted additive model as defined by Lovell and Pastor [19]. Additionally, since the multiplier form is the linear dual model of envelopment form, we have that $L^*(x_0, y_0; LNC4) = \sum_{i=1}^m w_i^- s_{i0}^- + \sum_{r=1}^s w_r^+ s_{r0}^+$, i.e., the weighted additive model, measures inefficiency.

3.3. Russell Models

Model 5. The input-oriented Russell measure

Let us consider the linear loss function model with our fifth linear normalization condition, $LNC5$. It assumes that $x_0 > 0_m$.

$$\begin{aligned}
L^*(x_0, y_0; LNC5) = & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
& \text{s.t.} \\
& \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
& c \geq 0_m, p \geq 0_s \\
& c_i x_{i0} = \frac{1}{m}, i = 1, \dots, m \quad (LNC5)
\end{aligned}$$

The dual of this model, after adjusting the lambda's, is

$$\begin{aligned}
& \text{Max} && \frac{1}{m} \sum_{i=1}^m \tau_i \\
& \text{s.t.} && \\
& && \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} (1 - \tau_i), \quad i = 1, \dots, m \\
& && \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad r = 1, \dots, s \\
& && \sum_{j=1}^n \lambda_j = 1 \\
& && \lambda_j \geq 0, \quad j = 1, \dots, n \\
& && \tau_i \in \mathbb{R}, \quad i = 1, \dots, m
\end{aligned}$$

This model is equivalent (in its optimal solutions) to considering a further model with the same restrictions and an objective function equals to $\text{Min} \left\{ 1 - \frac{1}{m} \sum_{i=1}^m \tau_i \right\}$. Now, we perform the following change of variables $\theta_i = 1 - \tau_i$, $i = 1, \dots, m$, obtaining

$$\begin{aligned}
& \text{Min} && \frac{1}{m} \sum_{i=1}^m \theta_i \\
& \text{s.t.} && \\
& && \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{i0}, \quad i = 1, \dots, m \\
& && \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad r = 1, \dots, s \\
& && \sum_{j=1}^n \lambda_j = 1 \\
& && \lambda_j \geq 0, \quad j = 1, \dots, n \\
& && \theta_i \in \mathbb{R}, \quad i = 1, \dots, m
\end{aligned}$$

Since $\lambda_j = 0, j \neq 0, \lambda_0 = 1, \theta_i = 1, i = 1, \dots, m$ is a feasible solution of the above model, we have that $0 \leq \theta_i^* \leq 1$.

Observe that this model coincides with the input-oriented Russell measure of Färe and Lovell [26]. It corresponds to the envelopment form of the model.

Finally, if we consider the changes we have made, we can relate the optimal loss function with the input oriented Russell measure. In fact,

$$1 - L^*(x_0, y_0; LNC5) = \frac{1}{m} \sum_{i=1}^m \theta_i^* .$$

Model 6. The output-oriented Russell measure

Let us consider the linear loss function model with our sixth linear normalization condition, $LNC6$. It assumes that $y_0 > 0_s$.

$$\begin{aligned}
 L^*(x_0, y_0; LNC6) = \quad & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 \text{s.t.} \quad & \\
 & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
 & c \geq 0_m, p \geq 0_s \\
 & p_r y_{r0} = \frac{1}{s}, r = 1, \dots, s \quad (LNC6)
 \end{aligned}$$

The dual of this model, after adjusting the lambda's, is:

$$\begin{aligned}
 \text{Max} \quad & \frac{1}{s} \sum_{r=1}^s \tau_r \\
 \text{s.t.} \quad & \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} (1 + \tau_r), \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & \tau_r \in R, \quad r = 1, \dots, s
 \end{aligned}$$

This model is equivalent (in its optimal solutions) to considering a further model with the same restrictions and an objective function equal to $Max \left\{ 1 + \frac{1}{s} \sum_{r=1}^s \tau_r \right\}$.

Now, we perform the change of variables $\phi_r = 1 + \tau_r$, $r = 1, \dots, s$, obtaining

$$\begin{aligned}
 &Max \quad \frac{1}{s} \sum_{r=1}^s \phi_r \\
 &s.t. \\
 &\quad \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \dots, m \\
 &\quad \sum_{j=1}^n \lambda_j y_{rj} \geq \phi_r y_{r0}, \quad r = 1, \dots, s \\
 &\quad \sum_{j=1}^n \lambda_j = 1 \\
 &\quad \lambda_j \geq 0, \quad j = 1, \dots, n \\
 &\quad \phi_r \in R, \quad r = 1, \dots, s
 \end{aligned}$$

Since $\lambda_j = 0$, $j \neq 0$, $\lambda_0 = 1$, $\phi_r = 1$, $r = 1, \dots, s$ is a feasible solution of the above model, we have that $\phi_r^* \geq 1$, $r = 1, \dots, s$.

Observe that this model coincides with the output-oriented Russell measure of Färe and Lovell [26]. It corresponds with the envelopment form of the model. Finally, if we consider the changes we have made, we can relate the optimal loss function with the output oriented Russell measure. In fact, following the

changes we have made we get $1 + L^*(x_0, y_0; LNC6) = \frac{1}{s} \sum_{r=1}^s \phi_r^*$.

Model 7. The enhanced Russell graph measure

Let us consider the linear loss function model with our seventh linear normalization condition, $LNC7$. It assumes that $x_0 > 0_m$ and $y_0 > 0_s$.

$$\begin{aligned}
L^*(x_0, y_0; LNC7) = & \text{Min} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
& \text{s.t.} \\
& \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad j = 1, \dots, n \\
& c \geq 0_m, p \geq 0_s \\
& \left. \begin{aligned} & c_i x_{i0} \geq \frac{1}{m}, i = 1, \dots, m \\ & p_r y_{r0} \geq \frac{1}{s} \left(1 + \sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m c_i x_{i0} - \alpha \right), r = 1, \dots, s \end{aligned} \right\} (LNC7)
\end{aligned}$$

This model is equivalent (in its optimal solutions) to considering a further model with the same constraints and an objective function equal to

$$\text{Max} \left\{ 1 - \left(-\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \right) \right\}.$$

Now, let us perform the following change of variables $\omega = 1 - \left(-\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \right)$. Then the equivalent (in its optimal

solutions) reformulation of this model is

$$\text{Max} \quad \omega$$

s.t.

$$\begin{aligned}
\omega &= 1 - \sum_{i=1}^m c_i x_{i0} + \sum_{r=1}^s p_r y_{r0} - \alpha \\
-\sum_{i=1}^m c_i x_{ij} + \sum_{r=1}^s p_r y_{rj} - \alpha &\leq 0, \quad j = 1, \dots, n \\
-c_i &\leq -\frac{1}{m \cdot x_{i0}}, \quad i = 1, \dots, m \\
\frac{\omega}{s \cdot y_{r0}} - p_r &\leq 0, \quad r = 1, \dots, s \\
c &\geq 0_m, p \geq 0_s
\end{aligned}$$

The first added restriction is just the definition of ω . The last set of restrictions has been reordered so as to have all the variables on the same side. Then, the linear dual of the last model is

$$\begin{aligned}
\text{Min } & \beta - \frac{1}{m} \sum_{i=1}^m \frac{t_{i0}^-}{x_{i0}} \\
\text{s.t. } & \\
& \beta + \frac{1}{s} \sum_{r=1}^s \frac{t_{r0}^+}{y_{r0}} = 1 \\
& \beta x_{i0} - \sum_{j=1}^n \mu_j x_{ij} - t_{i0}^- \geq 0, \quad i = 1, \dots, m \\
& -\beta y_{r0} + \sum_{j=1}^n \mu_j y_{rj} - t_{r0}^+ \geq 0, \quad r = 1, \dots, s \\
& \beta - \sum_{j=1}^n \mu_j = 0, \\
& \mu \geq 0_n, t_0^- \geq 0_m, t_0^+ \geq 0_s
\end{aligned}$$

Making the following change of variables $t_{i0}^- = \beta s_{i0}^-$, $i = 1, \dots, m$, $t_{r0}^+ = \beta s_{r0}^+$, $r = 1, \dots, s$, $\mu_j = \beta \lambda_j$, $j = 1, \dots, n$, we get

$$\begin{aligned}
\text{Min } & \beta \left(1 - \frac{1}{m} \sum_{i=1}^m \frac{s_{i0}^-}{x_{i0}} \right) \\
\text{s.t. } & \\
& \beta \left(1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{y_{r0}} \right) = 1 \\
& \beta \left(x_{i0} - \sum_{j=1}^n \lambda_j x_{ij} - s_{i0}^- \right) \geq 0, \quad i = 1, \dots, m \\
& -\beta \left(y_{r0} - \sum_{j=1}^n \lambda_j y_{rj} + s_{r0}^+ \right) \geq 0, \quad r = 1, \dots, s \\
& \beta \left(1 - \sum_{j=1}^n \lambda_j \right) = 0, \\
& \lambda \geq 0_n, s_0^- \geq 0_m, s_0^+ \geq 0_s
\end{aligned}$$

The first restriction tells us two things. First that $\beta = \left(1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{y_{r0}} \right)^{-1} > 0$, which means that the objective function can be rewritten as shown below, and second, as a consequence, that the set of restrictions but the first one can be simplified by deleting β . Therefore this nonlinear program can be rewritten as

$$\text{Min} \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{s_{i0}^-}{x_{i0}}}{1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{y_{r0}}}$$

s.t.

$$x_{i0} - \sum_{j=1}^n \lambda_j x_{ij} - s_{i0}^- \geq 0 \quad i = 1, \dots, m$$

$$y_{r0} - \sum_{j=1}^n \lambda_j y_{rj} + s_{r0}^+ \leq 0 \quad r = 1, \dots, s$$

$$1 - \sum_{j=1}^n \lambda_j = 0,$$

$$\lambda \geq 0_n, s_0^- \geq 0_m, s_0^+ \geq 0_s$$

As can be seen, the restrictions are exactly the restrictions of the additive model. As is well known, the two initial set of restrictions can be equivalently written as equalities. Therefore, if we perform the change of variables

$\theta_i = 1 - \frac{s_{i0}^-}{x_{i0}}, i = 1, \dots, m, \phi_r = 1 + \frac{s_{r0}^+}{y_{r0}}, r = 1, \dots, s$, we finally get

$$\text{Min} \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{r=1}^s \phi_r}$$

s.t.

$$\sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{i0}, \quad i = 1, \dots, m$$

$$\sum_{j=1}^n \lambda_j y_{rj} \geq \phi_r y_{r0}, \quad r = 1, \dots, s$$

$$\sum_{j=1}^n \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j = 1, \dots, n$$

$$\theta_i \leq 1, \quad i = 1, \dots, m$$

$$\phi_r \geq 1, \quad r = 1, \dots, s$$

which is, exactly, the enhanced Russell graph model of Pastor et al. [27]). The last model is also known in the literature as the SBM (Slacks-Based Measure) model (Tone [28]).

Finally, observe that considering all the above steps, we have at optimum

$$1 - L^*(x_0, y_0; LNC7) = \frac{\frac{1}{m} \sum_{i=1}^m \theta_i^*}{\frac{1}{s} \sum_{r=1}^s \phi_r^*}.$$

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