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Families of Linear Efficiency Programs based
on Debreu's Loss Function

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Abstract

Gerard Debreu introduced a well known radial efficiency measure which he called a "coefficient of resource utilization." He derived this scalar from a much less well known "dead loss" function that characterizes the monetary value sacrificed to inefficiency, and which is to be minimized subject to a normalization condition. We use Debreu's loss function, together with a variety of normalization conditions, to generate several popular families of linear efficiency programs. Our methodology also can be employed to generate entirely new families of linear efficiency programs.

Key words: loss function; linear efficiency programs; normalization conditions; DEA

JEL Codes: C51, C61

1. Introduction

The famous paper by Debreu (1951), “The Coefficient of Resource Utilization,” has inspired this study. Farrell (1957, pp. 253-54) remarked that “The professional economist...can note the similarity of the measure of ‘technical efficiency’ and Debreu’s ‘coefficient of resource utilization’,” although in our opinion the similarity has been exaggerated. However the concept of Debreu that has most influenced this study is his loss function, which Farrell did not mention, and which has gone largely overlooked in the literature¹. This concept, which Debreu initially developed to evaluate the “dead loss” associated with a non-optimal allocation of resources in an economic system, is a money metric measure of the distance from an actual allocation to a set of optimal allocations, i.e., “the minimum of the distance from the given complex to a varying optimal complex.” After proving “the intrinsic existence of price systems associated with the optimal complexes of physical resources,” the minimization problem proposed by Debreu is $Min_z p_z \cdot (z_0 - z)$, with z_0 a vector representing the actual allocation of resources, z a vector belonging to the set of optimal allocations and p_z one of the corresponding shadow, or intrinsic, price vectors. Debreu named the optimal value of this problem “the magnitude of the loss”, and he proved that $p_z \cdot (z_0 - z) \geq 0$, recognizing that “ p_z is affected by an arbitrary positive multiplicative scalar”. The influence of this scalar means that the magnitude of the loss can be driven to zero by an appropriate scaling of all elements of p_z . “In order to eliminate the arbitrary multiplicative factor affecting all the prices,” Debreu proposed to divide the objective function by a price index, either $p_z \cdot z_0$ or $p_z \cdot z$, reformulating the original problem as $Min_z p_z \cdot (z_0 - z) / p_z \cdot z_0$, or, equivalently, $Max_z p_z \cdot z / p_z \cdot z_0$.

¹ Diewert (1983) extended Debreu’s loss measure, but in a different context and in a different way than we do. Diewert focused his analysis on measuring the output loss that can be attributed to distortions within the production sector of an open economy. In addition, Diewert did not consider alternative normalization conditions as we do.

It is clever to show, as Debreu did, that an optimal solution to the maximization problem is $z^* = \rho z_0$, where the scalar $\rho \leq 1$ is Debreu's "coefficient of resource utilization." Moreover, $\rho = 1$ if, and only if, the actual allocation z_0 belongs to the set of optimal allocations (i.e., is efficient), and $\rho < 1$ if, and only if, the actual allocation is feasible but not efficient.

The minimization problem is nonlinear in both variables p_z and z . We stress that it is not compulsory to resort to a normalization factor, because the influence of the arbitrary multiplicative scalar can also be eliminated by adding restrictions to the loss minimization problem. In fact, Debreu's problem can be rewritten as

$$\begin{array}{l} \text{Min}_z \quad p_z \cdot (z_0 - z) \\ \text{s.t.} \quad p_z \cdot z_0 = 1 \end{array}$$

Neither Debreu's normalization condition nor our added restriction is unique. In addition, the normalization condition involves all the intrinsic prices, just as the normalization factor of Debreu does.

Debreu studied an economic system consisting of two activities, production and consumption, and having three sources of loss, underemployment of resources, inefficiency in production and imperfection of economic organization. We simplify matters by studying the production activity of an economic system having one source of loss, which Debreu calls "the technical inefficiency of production units."^{2,3}

In a production context Debreu's economic sector resources vector z narrows to a production sector quantity vector of inputs and outputs, and p_z is a vector of their respective prices. In this context we can use the loss function minimization method introduced by Debreu to evaluate the technical efficiency of any

² The above quoted phrases are from Debreu (1951, pp. 274, 275, 284).

³ ten Raa (2008) provides a discussion of Debreu's economic system, part of which is our production sector.

producer, assuming that the optimal producers have intrinsic prices affected by a positive scalar unless a normalization scheme is introduced. In our case, the existence of nonnegative intrinsic prices is guaranteed by the structure we impose on the production set. Moreover these assumptions also allow us to simplify our initially designed program by eliminating some of its variables.

The paper unfolds as follows. In section 2 we list the requirements that the production set must satisfy, and we formulate an initial version of our loss function minimization program. This version of the program is formulated in a generic way because the restrictions relating the set of intrinsic prices to the corresponding optimal allocation are not formulated mathematically, and so the normalization condition is not explicitly specified. This program seeks, similar to Debreu's method, the minimum of the distance from the production unit under evaluation to a varying optimal allocation in the production set, and depends both on the optimal allocation and on its intrinsic prices. We then obtain a second version of the program, equivalent to the initial version, which linearizes the objective function of the loss minimization program, and characterizes the geometric nature of the program as a supporting hyperplane program. It does so by eliminating the optimal allocation from the minimization problem, which depends only on the set of intrinsic prices and the intercept of the supporting hyperplane. In section 3 we further specialize the second version of the program by introducing a common set of mathematical restrictions, but with a sequence of different normalization conditions, giving rise to several well known families of efficiency programs that either are linear or can be linearized. Section 4 concludes.

2. The Loss Function

In this section we define the loss function in a production context. To this end we introduce some notation. A vector of m inputs is denoted by $x = (x_1, \dots, x_m)$ and a vector of s outputs is denoted by $y = (y_1, \dots, y_s)$. A vector of m input prices is denoted by $c = (c_1, \dots, c_m)$ and a vector of s output prices is denoted by

$p = (p_1, \dots, p_s)$. The production technology is given by the set $T = \{(x, y) : x \in R_+^m \setminus \{0_m\}, y \in R_+^s \setminus \{0_s\}, x \text{ can produce } y\}$. We assume that T is nonempty (P1), closed (P2), convex (P3) and satisfies strong disposability (P4).

Not all input-output vectors belonging to the production technology are technically efficient. To measure efficiency it is necessary to compare actual performance with a subset of the boundary of T , defined as follows.

Definition 1. The weakly efficient subset of T , $\partial^W(T)$, is defined as

$$\partial^W(T) = \{(x, y) \in T : (-u, v) > (-x, y) \Rightarrow (u, v) \notin T\}^4$$

Postulate P3 guarantees existence of a supporting hyperplane at any frontier point, and P4 guarantees that all $(x, y) \in \partial^W(T)$ satisfy a weak version of the Koopmans (1951) efficiency condition. In the context of Definition 1, the strict inequality provides a weak version of Koopmans' efficiency condition, while the weak inequality provides a strong version. Both are consistent with P4.

We think of the firm as a competitive profit maximizer, taking prices as fixed and choosing a feasible production plan $(x, y) \in T$ which maximizes its profit. The resulting (optimum) profit is a function of the price vector (c, p) which we denote by $\Pi(c, p)$.

Definition 2. Given a vector of input and output prices $(c, p) \in R_+^m \times R_+^s$, and a production technology T , the firm's profit function Π is defined as

$$\Pi(c, p) = \max_{x, y} \left\{ \sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i : (x, y) \in T \right\}.$$

⁴ $(a, b) > (d, e)$ means that $a_i > d_i, \forall i = 1, \dots, m$ and $b_r > e_r, \forall r = 1, \dots, s$. $(a, b) \geq (d, e)$ means that $a_i \geq d_i, \forall i = 1, \dots, m$ and $b_r \geq e_r, \forall r = 1, \dots, s$.

Postulates P1-P4 establish a duality between the profit function Π and the production technology T , with T recovered by (see Färe and Primont (1995))

$$T = \left\{ (x, y) \in R_+^m \times R_+^s : \sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \leq \Pi(c, p), \forall (c, p) \in R_+^m \times R_+^s \right\}.$$

Applying the supporting hyperplane theorem and invoking P2, P3 and P4, for each $(x, y) \in \partial^W(T)$ there exists at least one shadow, or intrinsic, price vector

$$(c, p) \in R_+^m \times R_+^s \text{ such that } (u, v) \in T \text{ implies } \sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \geq \sum_{r=1}^s p_r v_r - \sum_{i=1}^m c_i u_i.$$

And from Definition 2, $\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i = \Pi(c, p)$. Therefore, the vector $(c, p, \Pi(c, p))$ defines a supporting hyperplane⁵ of T .

We denote hereafter the set of all shadow prices of $(x, y) \in \partial^W(T)$ by $Q(x, y)$. Also, we denote the set of all the vectors $(c, p, \alpha) \in R_+^m \times R_+^s \times R$ which define a supporting hyperplane of T by $SH(T)$.

We will define a loss function minimization program inspired by Debreu's problem but containing a broader set of normalization conditions. More specifically, we will consider a normalization condition involving multiple restrictions, not all of which involve all shadow prices. One important difference between our approach and Debreu's is that Debreu considered alternative normalizations which lead to the same solution, while we will consider a wider family of normalization conditions that generate different loss function minimization programs and different inefficiency measures.

⁵ Given a vector $(\tilde{x}, \tilde{y}) \in R_+^m \times R_+^s$, a vector $(c, p, \alpha) \in R_+^m \times R_+^s \times R$ defines a hyperplane given by the equation $\sum_{r=1}^s p_r \tilde{y}_r - \sum_{i=1}^m c_i \tilde{x}_i = \alpha$. By definition, a supporting hyperplane of T is a hyperplane that contains at least one point of $\partial^W(T)$, and $\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \leq \alpha$, for all $(x, y) \in T$.

We are now prepared to introduce the main concept of the paper: the loss function.

Definition 3. Given $(x_0, y_0) \in R_+^m \times R_+^s$ and NC , a set of constraints on the shadow prices, the loss function $L(x_0, y_0; NC)$ is defined as the optimal value of the minimization program A1:

$$\begin{aligned}
 L(x_0, y_0; NC) := & \min_{x, y, c, p} \sum_{i=1}^m c_i(x, y)(x_{i0} - x_i) + \sum_{r=1}^s p_r(x, y)(y_r - y_{r0}) \\
 \text{s.t.} & (x, y) \in \partial^W(T), (c(x, y), p(x, y)) \in Q(x, y) \\
 & NC(c(x, y), p(x, y))
 \end{aligned} \tag{A1}$$

We refer to $L(x_0, y_0; NC)$ as the loss function corresponding to allocation (x_0, y_0) for the normalization condition $NC(c(x, y), p(x, y))$, which can be expressed through any number of restrictions. We assume that the set of shadow prices that satisfies $NC(c(x, y), p(x, y))$ is non-empty and closed, and that the null vector does not satisfy $NC(c(x, y), p(x, y))$, to guarantee that the shadow prices are not affected by an arbitrary positive multiplicative scalar.

The objective function in program A1 is a nonlinear function of an optimal allocation and its shadow prices, and, a priori, difficult to solve. We next develop an equivalent formulation that has a linear objective function whose shadow prices do not depend on $(x, y) \in \partial^W(T)$.

First of all, if $(c(x, y), p(x, y))$ is a shadow price vector of $(x, y) \in \partial^W(T)$ then

$$\sum_{r=1}^s p_r(x, y)y_r - \sum_{i=1}^m c_i(x, y)x_i = \Pi(c(x, y), p(x, y)), \text{ as we established above.}$$

Therefore the following program, A2, is equivalent to the loss function minimization program A1.

$$\begin{aligned}
L(x_0, y_0; NC) = & \min_{x, y, c, p} \Pi(c(x, y), p(x, y)) - \left(\sum_{r=1}^s p_r(x, y) y_{r0} - \sum_{i=1}^m c_i(x, y) x_{i0} \right) \\
\text{s.t.} & (x, y) \in \partial^W(T), (c(x, y), p(x, y)) \in Q(x, y) \\
& NC(c(x, y), p(x, y))
\end{aligned} \tag{A2}$$

Consequently, minimizing the difference between the profit function and profit at the actual allocation (x_0, y_0) , subject to constraints, yields the loss function.

Second, we are able to obtain the loss function $L(x_0, y_0; NC)$ by means of a program with a linear objective function that eliminates the dependence of shadow prices on allocations belonging to the frontier of T , as Proposition 1 shows.

Proposition 1. Program A2 has the same optimal value as the following program A3:

$$\begin{aligned}
L(x_0, y_0; NC) = & \min_{c, p, \alpha} \alpha - \left(\sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m c_i x_{i0} \right) \\
\text{s.t.} & (c, p, \alpha) \in SH(T) \\
& NC(c, p)
\end{aligned} \tag{A3}$$

Proof. See Appendix.

The geometrical idea behind Proposition 1 is not hard to see. By duality, a convex technology has an alternative representation through supporting hyperplanes. As a result, instead of using information on all allocations in $\partial^W(T)$ and their corresponding shadow prices in programs A1 and A2, we can equivalently use information on all supporting hyperplanes of T in program A3.

Henceforth we will use program A3 to calculate the loss function. In this particular program α^* is shadow profit $\Pi(c^*, p^*)$, where (c^*, p^*, α^*) is an optimal solution of program A3.

3. Deriving Families of Linear Efficiency Programs

We restrict our analysis to either linear efficiency programs or nonlinear efficiency programs that can be linearized. In either case, in the linear loss function program we impose linearity on the normalization condition, so that it can be represented by means of a finite set of equalities and/or inequalities which are linear in (c, p, α) , and we write $LNC(c, p)$ instead of $NC(c, p)$. Moreover, from now on we assume that T is constructed from a finite set of n homogeneous production units $\{(x_j, y_j), j = 1, \dots, n\}$, as any DEA (Data Envelopment Analysis) efficiency program does. In this way, T is defined precisely as

$$T = \left\{ (x, y) \in R_+^m \times R_+^s : (x, -y) \geq \sum_{j=1}^n \lambda_j (x_j, -y_j), \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\},$$

which allows the technology to satisfy variable returns to scale (VRS). If we want to force the technology to satisfy non-increasing returns to scale (NIRS), non-decreasing returns to scale (NDRS) or constant returns to scale (CRS), we simply modify the convexity constraint on the sum of the intensity variables λ_j in

the definition of T as follows: *NIRS*: $\sum_{j=1}^n \lambda_j \leq 1$; *NDRS*: $\sum_{j=1}^n \lambda_j \geq 1$; *CRS*:

$\sum_{j=1}^n \lambda_j \geq 0$. From now on and for the sake of brevity we will only deal with the

VRS case. Obviously, any other case can be formulated similarly. We include NDRS, which includes CRS, for completeness, even though NDRS is incompatible with *price taking* profit maximization.

Consider the next linear loss function program, A4, to evaluate the efficiency of a production unit (x_0, y_0) belonging to $\{(x_j, y_j), j = 1, \dots, n\}$.

$$\begin{aligned}
L(x_0, y_0; LNC) = & \min_{c, p, \alpha} -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
\text{s.t.} & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\
& c \geq 0_m, p \geq 0_s \\
& LNC(c, p)
\end{aligned} \tag{A4}$$

Proposition 2. The linear loss function program A4 has the same optimal value and optimal solutions as the loss function program A3.

Proof. See Appendix.

We do not need to declare in program A4 that the set of hyperplanes we are considering are supporting hyperplanes of T , i.e., $(c, p, \alpha) \in SH(T)$, because the minimization process does the job for us.

Since in the linear loss function program A4, α is free, the supporting hyperplane has an intercept unrestricted in sign. This corresponds to the VRS technology specified above, and we restrict our subsequent analysis to VRS programs. These are the DEA programs that are most similar to Debreu's formulation. Nevertheless, in a DEA framework, the generation of loss function programs under alternative returns to scale assumptions is straightforward. For a NIRS program we add to the above program the restriction $\alpha \geq 0$; for a NDRS program we add the restriction $\alpha \leq 0$; and for a CRS program we add the restriction $\alpha = 0$ or, equivalently, we delete α everywhere.

3.1. Radial DEA Programs and Directional Distance Function Programs

Radial DEA programs have evolved from Debreu's coefficient of resource utilization and Farrell's measure of technical efficiency, and involve *scaling* observed quantity vectors. Directional distance function programs have evolved from Debreu's loss function and Luenberger's (1992a, 1992b) benefit and shortage functions, and involve *translating* observed quantity vectors. We consider both types of program, as well as extensions of both. We provide a

new unifying way of dealing with both programs, exploring the structure of the multiplier form, rather than the more popular envelopment form. The only difference between any pair of VRS-DEA programs, from a mathematical point of view, is the finite set of restrictions we call normalization conditions, as we show next. In addition, in the objective function it is possible to conduct the minimization over fewer than m inputs and/or fewer than s outputs. Such a framework corresponds to a money metric measure of sub-vector efficiency, or of efficiency in the presence of non-discretionary or quasi-fixed variables. We do not highlight this possibility, but we remind the reader of its existence.

Program 1. The BCC (Banker et al. (1984)) input-oriented program

We write the linear duals of the BCC input-oriented program as

Envelopment form

$$\begin{aligned} \min_{\lambda, \theta} \quad & \theta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{i0}, \quad \forall i \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad \forall r \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda \geq 0_n \end{aligned}$$

Multiplier form

$$\begin{aligned} \max_{c, p, \alpha} \quad & \sum_{r=1}^s p_r y_{r0} - \alpha \\ \text{s.t.} \quad & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\ & \sum_{i=1}^m c_i x_{i0} = 1 \\ & c \geq 0_m, p \geq 0_s \end{aligned}$$

A comparison of the multiplier form and the linear loss function program A4 suggests that the corresponding linear normalization condition is

$$\sum_{i=1}^m c_i x_{i0} = 1 \quad (LNC1).$$

In this input-oriented program the normalization condition involves input prices only. It is easy to show (see Appendix) that by manipulating the corresponding linear loss function program, we have at optimum

$$1 - L(x_0, y_0; LNC1) = \sum_{r=1}^s p_r^* y_{r0} - \alpha^*, \text{ i.e., the optimal value of the multiplier form of}$$

the BCC input-oriented program. Since the minimized loss function measures inefficiency, the maximized objective of the BCC program measures efficiency.

If the rated unit is efficient, then $\sum_{r=1}^s p_r^* y_{r0} - \alpha^* = 1 \Leftrightarrow L(x_0, y_0; LNC1) = 0$.

Debreu distinguished the magnitude of the loss from the value of the loss. The value of the loss can be determined by multiplying the envelopment form objective and the multiplier form linear normalization condition by actual cost C_0 . This yields the money metric value of the loss as $C_0(1 - \theta^*) = C_0 L(x_0, y_0; LNC1)$.

Related programs. It is apparent that the CCR (Charnes et al. (1978)) input-oriented linear program is also related to the loss function. On the other hand, the nonlinear CRS hyperbolic program of Färe et al. (1985) is related to the loss function because it can be linearized to the CCR input-oriented program. The AR (Assurance Region) input-oriented programs, introduced by Thompson et al. (1986), are all related to the CCR input-oriented program in the following sense. The multiplier form of an AR program includes all the restrictions of the CCR program, and the same linear normalization condition, together with a set of “value judgment” restrictions such as $\left\{ \frac{c_i}{c_1} \geq k_i, i = 2, \dots, m \right\}$, $k_i > 0$, $i = 1, \dots, m$.

Our linear loss function program can be easily extended to cover AR programs, just by adding the same value judgment restrictions.

Program 2. The BCC output-oriented program

We write the linear duals of the BCC output-oriented program as

Envelopment form

$$\begin{aligned}
 & \max_{\lambda, \phi} \quad \phi \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad \forall i \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \phi y_{r0}, \quad \forall r \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda \geq 0_n
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \min_{c, p, \alpha} \quad \sum_{i=1}^m c_i x_{i0} + \alpha \\
 \text{s.t.} \quad & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\
 & \sum_{r=1}^s p_r y_{r0} = 1 \\
 & c \geq 0_m, p \geq 0_s
 \end{aligned}$$

The corresponding linear normalization condition normalizes output prices only,

$$\sum_{r=1}^s p_r y_{r0} = 1 \quad (LNC2),$$

and by manipulating the linear loss function program with *LNC2*, we have at

optimum $1 + L(x_0, y_0; LNC2) = \sum_{i=1}^m c_i^* x_{i0} + \alpha^*$, which specifies the relationship

between the minimized loss function and the optimal objective function of the BCC output-oriented program. For any efficient point,

$$\sum_{i=1}^m c_i^* x_{i0} + \alpha^* = 1 \Leftrightarrow L(x_0, y_0; LNC2) = 0.$$

The money metric value of any loss is evaluated as in Program 1, multiplying the envelopment form objective and the multiplier form linear normalization condition by actual revenue R_0 .

Program 3. The directional distance function program

We write the dual forms of the directional distance function program, with

$g = (g^-, g^+) \in R_+^m \times R_+^s$ a pre-specified non-null directional vector, as

Envelopment form

$$\begin{aligned}
 & \max_{\lambda, \beta} \quad \beta \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} - \beta g_i^-, \quad \forall i \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} + \beta g_r^+, \quad \forall r \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda \geq 0_n
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \min_{c, p, \alpha} \quad -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 \text{s.t.} \quad & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\
 & \sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c_i g_i^- = 1, \\
 & c \geq 0_m, p \geq 0_s
 \end{aligned}$$

In this program the corresponding linear normalization condition normalizes both p and c by means of

$$\sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c_i g_i^- = 1 \quad (LNC3) .$$

Exploiting the dual of the linear loss function program with $LNC3$, we have at optimum $L(x_0, y_0; LNC3) = \beta^*$, which describes the relationship between the loss function and the shortage function of Luenberger (1992b), and which Chambers et al. (1998) refer to as the directional technology distance function⁶. Both functions provide a measure of inefficiency, and so $\beta^* = 0 \Leftrightarrow L(x_0, y_0; LNC3) = 0$.

Related programs. Constraining the directional vector to either $g = (g^-, 0_s)$ or $g = (0_m, g^+)$ generates two special cases of Luenberger's shortage function, which Chambers et al. (1996, 1998) refer to as directional input and output distance functions, respectively. Programs 1 and 2 can indirectly be obtained as

⁶ Chambers et al. (1998) prove that there is a dual relationship between the profit function and the directional distance function. In particular, the directional distance function can be recovered from the profit function by means of $\beta^* =$

$\inf_{(c, p) \geq (0_m, 0_s)} \left\{ \Pi(c, p) - \left(\sum_{r=1}^s p_r y_r - \sum_{i=1}^m c_i x_i \right) : \sum_{r=1}^s p_r g_r^+ + \sum_{i=1}^m c_i g_i^- = 1 \right\}$. This is clearly a particular case of program A2 taking as NC the linear condition $LNC3$, since $\Pi(c, p)$ is defined only for prices that support points belonging to $\partial^W(T)$.

particular cases, by setting $g = (x_0, 0_s)$ to generate the inefficiency associated with the BCC input-oriented program, and choosing $g = (0_m, y_0)$ to perform the same task for the BCC output-oriented program. Bric's (1999) ℓ_∞ distance to $\partial^W(T)$ is obtained by setting $g = (1_m, 1_s)$ ⁷. Additional data-dependent programs have been developed: The Range Directional Program (RDM) of Silva Portela et al. (2004), which considers an ideal allocation associated with the set of n units, defined as $z_i^- = \min_j \{x_{ji}\}, i = 1, \dots, m$, $z_r^+ = \max_j \{y_{jr}\}, r = 1, \dots, s$, so as to define the data-dependent directional vector $g_i^- = x_{i0} - z_i^-$, $i = 1, \dots, m$, $g_r^+ = z_r^+ - y_{r0}$, $r = 1, \dots, s$; the MEA program of Bogetoft and Hougaard (1999), which is also data dependent⁸; and Bric's (1997) "Graph-type extension of Farrell technical efficiency measure", which is another directional distance function program, obtained from the directional vector $g = (x_0, y_0)$.

3.2. Additive DEA Programs

We consider the weighted additive program of Lovell and Pastor (1995), which has the same restrictions as the additive program of Charnes et al. (1985), but its objective function is modified through the assignment of weights $(w^-, w^+) \in R_+^m \times R_+^s$ to the input slacks and the output slacks. The weights can vary across production units. This allows us to generate a wide range of additive programs in a unified way.

Program 4. The weighted additive program

The envelopment and multiplier forms of the weighted additive program are

⁷ Strictly speaking, the ℓ_∞ distance from (x_0, y_0) to $\partial^W(T)$ is equal to the directional distance function associated with the directional vector $g = (1_m, 1_s)$ only if $(x_0, y_0) \in T$; otherwise, the directional distance function is equal to $-[\text{the } \ell_\infty \text{ distance}]$.

⁸ Asmild and Pastor (2010) provide a detailed presentation of the RDM and MEA programs.

Envelopment form

$$\begin{aligned}
 & \max_{\lambda, s_0^-, s_0^+} \sum_{i=1}^m w_i^- s_{i0}^- + \sum_{r=1}^s w_r^+ s_{r0}^+ \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad \forall i \\
 & \quad \quad \sum_{j=1}^n \lambda_j y_{rj} = y_{r0} + s_{r0}^+, \quad \forall r \\
 & \quad \quad \sum_{j=1}^n \lambda_j = 1 \\
 & \quad \quad \lambda \geq 0_n, s_0^- \geq 0_m, s_0^+ \geq 0_s
 \end{aligned}$$

Multiplier form

$$\begin{aligned}
 & \min_{c, p, \alpha} -\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\
 & \quad \quad c \geq w^-, p \geq w^+
 \end{aligned}$$

The corresponding linear normalization condition is

$$\{c \geq w^-, p \geq w^+\} \text{ (LNC4) ,}$$

and by manipulating the linear loss function program with *LNC4*, we have at optimum $L(x_0, y_0; \text{LNC4}) = \sum_{i=1}^m w_i^- s_{i0}^{*-} + \sum_{r=1}^s w_r^+ s_{r0}^{*+}$, which specifies the relationship between the minimized loss function and the optimal objective function of the weighted additive program. The minimized objective function of the weighted additive program measures inefficiency, just as the minimized loss function does. Hence, for any efficient unit, $L(x_0, y_0; \text{LNC4}) = 0 \Leftrightarrow s_{i0}^{*-} = s_{r0}^{*+} = 0, \forall i = 1, \dots, m, \forall r = 1, \dots, s$. In this program the linear normalization condition is a set of linear inequalities; prices are bounded below by the weights attached to the slacks in the envelopment form program, without being uniquely determined.

Related programs. The CRS weighted additive program of Ali and Seiford (1993), the (standard) additive program of Charnes et al. (1985), which takes all weights equal to 1, the enhanced additive program of Charnes et al. (1987), also called MIP (Measure of Inefficiency Proportions) by Cooper et al. (1999),⁹

⁹ Based on the solution of the enhanced additive program, Bardhan et al. (1996) defined an efficiency measure called MED (Measure of Efficiency Dominance) which was renamed by Banker and Cooper (1994) as MEP (Measure of Efficiency Proportions), i.e., MEP=MED.

which takes $w_i^- = \frac{1}{x_{i0}}$, $i = 1, \dots, m$, $w_r^+ = \frac{1}{y_{r0}}$, $r = 1, \dots, s$ and requires all quantities to be strictly positive, are all related to the weighted additive program. The normalized weighted additive program of Lovell and Pastor (1995) is another related program which takes $w_i^- = \frac{1}{\sigma_i^-}$, $i = 1, \dots, m$, $w_r^+ = \frac{1}{\sigma_r^+}$, $r = 1, \dots, s$, where σ_i^- and σ_r^+ are the standard deviations of inputs and outputs over the n production units. The RAM (Range Adjusted Measure) of inefficiency program of Cooper et al. (1999) is yet another related program, taking $w_i^- = \frac{1}{(m+s)R_i^-}$, $i = 1, \dots, m$, $w_r^+ = \frac{1}{(m+s)R_r^+}$, $r = 1, \dots, s$, where R_i^- and R_r^+ are the ranges of inputs and outputs over the n production units. Cooper et al. (1999) define the corresponding measure of efficiency as 1-RAM. Finally, an improvement of the RAM has been recently proposed. It is known as BAM (Bounded Adjusted Measure) and requires to deal with range-bounded DEA models (see Cooper et al. (2010)).

3.3. Russell Programs

Russell programs were introduced, and named, by Färe and Lovell (1978) as a way of projecting, in a non-radial way, an observed allocation to the strongly efficient subset of technology.

Program 5. The input-oriented Russell program

The envelopment form of the input-oriented Russell program is

$$\begin{aligned}
\min_{\lambda, \theta} \quad & \frac{1}{m} \sum_{i=1}^m \theta_i \\
\text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} = \theta_i x_{i0}, \quad \forall i \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad \forall r \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \theta \leq \mathbf{1}_m, \\
& \lambda \geq \mathbf{0}_n
\end{aligned}$$

In order to show the relationship between Program 5 and the loss function we do not need to consider the multiplier form of the last program. We find it slightly easier to make a change of variables and to borrow the results obtained for Program 4 (see Appendix).

It is not apparent that in this program the corresponding linear normalization condition is

$$c_i \geq \frac{1}{mx_{i0}}, \quad i = 1, \dots, m \quad (\text{LNC5}).$$

Also, we have at optimum $1 - L(x_0, y_0; \text{LNC5}) = \frac{1}{m} \sum_{i=1}^m \theta_i^*$, which specifies the relationship between the minimized loss function and the optimal objective of the envelopment form. As with Program 1, the optimal value of Program 5 measures efficiency, in contrast to the loss function. In this case $L(x_0, y_0; \text{LNC5}) = 0 \Leftrightarrow \theta_i^* = 1, \forall i = 1, \dots, m$. Once again the linear normalization condition for this input-oriented program provides lower bounds for input prices.

Program 6. The output-oriented Russell program

The envelopment form of the output-oriented Russell program is

$$\begin{aligned}
& \max_{\lambda, \phi} \quad \frac{1}{s} \sum_{r=1}^s \phi_r \\
& \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad \forall i \\
& \quad \quad \sum_{j=1}^n \lambda_j y_{rj} = \phi_r y_{r0}, \quad \forall r \\
& \quad \quad \sum_{j=1}^n \lambda_j = 1 \\
& \quad \quad \phi \geq \mathbf{1}_s, \\
& \quad \quad \lambda \geq \mathbf{0}_n
\end{aligned}$$

The corresponding linear normalization condition provides lower bounds for output prices by means of

$$p_r \geq \frac{1}{s y_{r0}}, \quad r = 1, \dots, s \quad (\text{LNC6}).$$

Performing similar transformations as in Program 5, we have at optimum

$$1 + L(x_0, y_0; \text{LNC6}) = \frac{1}{s} \sum_{r=1}^s \phi_r^*,$$

which specifies the relationship between the minimized loss function and the optimal objective function of the output-oriented Russell program. As in Program 5, the optimal value measures efficiency, in contrast to the loss function, and so $L(x_0, y_0; \text{LNC6}) = 0 \Leftrightarrow \phi_r^* = 1, \forall r = 1, \dots, s$.

Program 7. The enhanced Russell graph program

The envelopment form of the enhanced Russell graph program, as defined in Pastor et al. (1999), is

$$\begin{aligned}
& \min_{\lambda, \theta, \phi} \quad \frac{1}{m} \sum_{i=1}^m \theta_i \Big/ \frac{1}{s} \sum_{r=1}^s \phi_r \\
& \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{i0}, \quad \forall i \\
& \quad \quad \sum_{j=1}^n \lambda_j y_{rj} \geq \phi_r y_{r0}, \quad \forall r \\
& \quad \quad \sum_{j=1}^n \lambda_j = 1 \\
& \quad \quad \lambda \geq 0_n, \theta \leq 1_m, \phi \geq 1_s
\end{aligned}$$

This model is nonlinear, although Pastor et al. (1999) showed that it can be linearized by means of a change of variables. Indeed, nonlinearity is the reason for considering only the envelopment form.

In this program the corresponding linear normalization condition is

$$\left\{ \begin{array}{l} c_i \geq \frac{1}{m x_{i0}}, i = 1, \dots, m \\ p_r \geq \frac{1}{s y_{r0}} \left(1 + \sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m c_i x_{i0} - \alpha \right), r = 1, \dots, s \end{array} \right\} \quad (LNC7) .$$

Let us observe that *LNC7* is the aggregation of *LNC5* and a modified version of *LNC6*. Hence *LNC7* provides lower bounds for both input prices and output prices.

Considering the dual of the linear loss function program with *LNC7*, and performing several transformations, we have at optimum (see the Appendix)

$$1 - L(x_0, y_0; LNC7) = \frac{1}{m} \sum_{i=1}^m \theta_i^* \Big/ \frac{1}{s} \sum_{r=1}^s \phi_r^* ,$$

which specifies the relationship between the minimized loss function and the objective function of the fractional form of the enhanced Russell graph program of Pastor et al. (1999). As in Programs 5 and 6, the optimal objective function

measures efficiency, in contrast to the loss function, and so $L(x_0, y_0; LNC7) = 0$

$$\Leftrightarrow \theta_i^* = \phi_r^* = 1, \forall i = 1, \dots, m, \forall r = 1, \dots, s.$$

3.4. A Hybrid Program

The idea of considering a multiplier form program that includes, at the same time, the restrictions *LNC1* and *LNC2* is due to Ray (2007). Mixing the input-oriented condition with the output-oriented condition gives rise to a “hybrid” program.

Program 8. The Shadow Profit Maximization Program

The envelopment and multiplier forms of the shadow profit maximization program of Ray (2007) are

Envelopment form

$$\begin{aligned} & \max_{\lambda, \theta, \phi} \quad \phi - \theta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{i0}, \quad \forall i \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq \phi y_{r0}, \quad \forall r \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda \geq 0_n, \phi \in R, \theta \in R \end{aligned}$$

Multiplier form

$$\begin{aligned} & \min_{c, p, \alpha} \quad \alpha \\ \text{s.t.} \quad & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\ & \left\{ \sum_{i=1}^m c_i x_{i0} = 1, \sum_{r=1}^s p_r y_{r0} = 1 \right\} \quad \forall r \\ & c \geq 0_m, p \geq 0_s \end{aligned}$$

A comparison of the multiplier form with the linear loss function program A4 suggests that the corresponding linear normalization condition is

$$\left\{ \sum_{i=1}^m c_i x_{i0} = 1, \sum_{r=1}^s p_r y_{r0} = 1 \right\} \quad (LNC8).$$

In this program the normalization condition involves the BCC input-oriented and output-oriented normalization conditions (see Programs 1 and 2). Consequently the linear normalization condition is expressed as a pair of linear equalities. It is

easy to show that by manipulating the corresponding linear loss function program, we have at optimum $L(x_0, y_0; LNC8) = \alpha^*$, which specifies that the optimal linear loss function equals the optimal value of the shadow profit maximization program.

4. Conclusions

Debreu's coefficient of resource utilization has attracted considerable attention through the years, but his dual loss function has been largely neglected. This oversight is unfortunate, and in this paper we demonstrate a new analytical use to which the loss function can be put. We narrow our focus from Debreu's economic system to its production activity, in which case the loss function provides a money metric measure of the value sacrificed to production inefficiency.

A generic loss function program, A1, appears in section 2. It is inspired by Debreu's formulation, and contains a nonlinear objective function, a feasibility condition, and a not-necessarily linear normalization condition. This program is completely linearized early in section 3, generating program A4. It contains a linear objective function, a set of linear inequalities describing feasibility conditions, and a linear normalization condition.

In section 3 we relate the linear loss function program A4 to several popular families of linear efficiency programs. In fact, this study seems to be the first systematic formulation for multiple efficiency measures in DEA. A structural feature of these programs is that all of them share the same subset of linear inequalities describing feasibility conditions and, in some cases, exactly the same linear objective function¹⁰. What varies across programs is the structure of the linear normalization condition. By varying this condition in predetermined

¹⁰ As Pastor and Aparicio (2010) have recently shown, linear programs that are associated with additive distance functions generate inefficiency measures (e.g., directional distance functions) and, as a consequence, have the same linear objective function as the corresponding linear loss function program. On the other hand, linear programs that are associated with multiplicative distance functions generate efficiency measures (e.g., BCC programs), and their objective functions are not the objective function of the corresponding linear loss function programs but are closely related to them.

ways we are able to derive all known DEA families of linear efficiency programs. Perhaps of greater value, we can vary this condition in new ways to generate new families of linear efficiency programs. Also, since all programs have the same structure, apart from the linear normalization condition, it is possible to conduct a uniform comparison of the abilities of each program to satisfy various desirable properties, such as units invariance (Lovell and Pastor (1995)) or translation invariance.

Thus an important implication of our analysis is that the derivation of linear efficiency programs need not be an ad hoc exercise. By resurrecting Debreu's loss function we have provided an analytical framework within which any, currently known or still unknown, linear efficiency program can be derived.

We conclude by highlighting two unresolved issues, the resolution of which might generate substantial benefits. The first concerns relations among programs. Portela and Thanassoulis (2006) claim that, under certain circumstances, weight restriction programs and non-radial programs are equivalent. This is a very strong claim. We refer to weight restriction radial programs in passing in our discussion of Program 1, and we discuss non-radial Russell programs in some detail in Programs 5-7. Under certain conditions, each reduce to Programs 1 or 2, but the relationship between the two sets of conditions is to be established. It is possible, but beyond the scope of this study, that the linear loss function program A4 can be put to another new use; it may provide an analytical framework within which to explore relationships between seemingly different programs.

The second unresolved issue concerns variation in the structure of the linear loss function. One aspect of structure is orientation. In all programs the prices being restricted correspond to the orientation of the program; input-oriented Programs 1 and 5 constrain input prices, output-oriented Programs 2 and 6 constrain output prices, and non-oriented Programs 3, 4, 7 and 8 constrain input prices and output prices.

A final, more intriguing, aspect of structure is the tautness with which the linear loss function constrains prices. In Programs 1-3 the linear loss function is a single equality, and in Program 8 it is a pair of equalities. However in Programs 4-7 the linear loss function is a system of linear inequalities. The causes and consequences of this differential tautness are worthy of further study.

APPENDIX

Proof of Proposition 1. Let (c^*, p^*, α^*) be an optimal solution of program A3.

Then, since $(c^*, p^*, \alpha^*) \in SH(T)$, there exists a vector $(x^*, y^*) \in \partial^W(T)$ such that

$$\sum_{r=1}^s p_r^* y_r^* - \sum_{i=1}^m c_i^* x_i^* = \alpha^* \quad \text{and} \quad \sum_{r=1}^s p_r^* y_r^* - \sum_{i=1}^m c_i^* x_i^* \geq \sum_{r=1}^s p_r^* v_r - \sum_{i=1}^m c_i^* u_i, \quad \forall (u, v) \in T.$$

Hence, by definition, $(c^*, p^*) \in Q(x^*, y^*)$. Now, we observe that

$(x^*, y^*; c(x^*, y^*), p(x^*, y^*))$, with $c(x^*, y^*) = c^*$ and $p(x^*, y^*) = p^*$, is a feasible

solution of A2. Finally, it is easy to prove that $(x^*, y^*; c(x^*, y^*), p(x^*, y^*))$ is also

an optimal solution of A2 and, in fact, program A2 has the same optimal value

as program A3. ■

Proof of Proposition 2. It is apparent from the structure of program A3 and the

fact that if $\sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j = 1, \dots, n$, then $\sum_{r=1}^s p_r v_r - \sum_{i=1}^m c_i u_i - \alpha \leq 0,$

$\forall (u, v) \in T$. Also, it is easy to prove that if (c^*, p^*, α^*) is an optimal solution of

program A4, then $(c^*, p^*, \alpha^*) \in SH(T)$. ■

We now prove Programs 1, 5 and 7. Proofs of the remaining programs are trivial.

Program 1. The BCC input-oriented program

Consider the linear loss function program A4 with linear normalization condition *LNC1*. As a consequence of *LNC1* the objective function of Program 1 is

equivalent to $1 + \min \left\{ -\sum_{r=1}^s p_r y_{r0} + \alpha \right\}$ and to $1 - \max \left\{ \sum_{r=1}^s p_r y_{r0} - \alpha \right\}$, which yields

$$\begin{aligned}
1-L(x_0, y_0; NC1) = & \max_{c, p, \alpha} \sum_{r=1}^s p_r y_{r0} - \alpha \\
\text{s.t.} & \sum_{r=1}^s p_r y_{rj} - \sum_{i=1}^m c_i x_{ij} - \alpha \leq 0, \quad \forall j \\
& c \geq 0_m, p \geq 0_s \\
& \sum_{i=1}^m c_i x_{i0} = 1 \quad (LNC1)
\end{aligned}$$

This program is exactly the multiplier form of the BCC input-oriented program. Being linear duals, the optimal value of the envelopment form equals the optimal value of the multiplier form. Therefore $1-L(x_0, y_0; LNC1) = \theta^*$.

Program 5. The input-oriented Russell program

This program assumes that $x_0 > 0_m$. By means of the change of variables

$$\theta_i = \frac{x_{i0} - s_{i0}^-}{x_{i0}} = 1 - \frac{s_{i0}^-}{x_{i0}}, \quad i = 1, \dots, m, \text{ we get that Program 5 is equivalent to}$$

$$\begin{aligned}
1- & \max_{\lambda, s_0^-} \sum_{i=1}^m \frac{s_{i0}^-}{m x_{i0}} \\
\text{s.t.} & \sum_{j=1}^n \lambda_j x_{ij} = x_{i0} - s_{i0}^-, \quad \forall i \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad \forall r \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda \geq 0_n, s_0^- \geq 0_m
\end{aligned}$$

In words, the input-oriented Russell program is equivalent to 1 minus a weighted additive program with weights $w_i^- = 1/mx_{i0}$, $i = 1, \dots, m$, and $w_r^+ = 0$, $r = 1, \dots, s$. Finally, thanks to Program 4, we have that $\{c_i \geq 1/mx_{i0}, i = 1, \dots, m\}$ are the normalization conditions for Program 5 and, at optimum,

$$1-L(x_0, y_0; LNC5) = \frac{1}{m} \sum_{i=1}^m \theta_i^* .$$

Program 7. The enhanced Russell graph program

Consider the linear loss function program A4 with linear normalization condition LNC7. This program assumes that $x_0 > 0_m$ and $y_0 > 0_s$. This program is equivalent (at its optimal solutions) to another program with the same constraints and objective function $\max \left\{ 1 - \left(-\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \right) \right\}$.

Performing the change of variables $\omega = 1 - \left(-\sum_{r=1}^s p_r y_{r0} + \sum_{i=1}^m c_i x_{i0} + \alpha \right)$ leads to the following equivalent reformulation (at its optimal solutions).

$$\begin{aligned}
 & \max_{c,p,\alpha,\omega} \quad \omega \\
 & \text{s.t.} \quad \omega = 1 - \sum_{i=1}^m c_i x_{i0} + \sum_{r=1}^s p_r y_{r0} - \alpha \\
 & \quad \quad -\sum_{i=1}^m c_i x_{ij} + \sum_{r=1}^s p_r y_{rj} - \alpha \leq 0, \quad \forall j \\
 & \quad \quad -c_i \leq -\frac{1}{m x_{i0}}, \quad \forall i \\
 & \quad \quad \frac{\omega}{s y_{r0}} - p_r \leq 0, \quad \forall r \\
 & \quad \quad c \geq 0_m, p \geq 0_s
 \end{aligned}$$

The first added restriction is just the definition of ω . The final set of restrictions has been reordered so as to have all the variables on the same side. The linear dual of the reformulated program is

$$\begin{aligned}
 & \min_{\beta,\mu,t_0^-,t_0^+} \quad \beta - \frac{1}{m} \sum_{i=1}^m \frac{t_{i0}^-}{x_{i0}} \\
 & \text{s.t.} \quad \beta + \frac{1}{s} \sum_{r=1}^s \frac{t_{r0}^+}{y_{r0}} = 1 \\
 & \quad \quad \beta x_{i0} - \sum_{j=1}^n \mu_j x_{ij} - t_{i0}^- \geq 0, \quad \forall i \\
 & \quad \quad -\beta y_{r0} + \sum_{j=1}^n \mu_j y_{rj} - t_{r0}^+ \geq 0, \quad \forall r \\
 & \quad \quad \beta - \sum_{j=1}^n \mu_j = 0, \\
 & \quad \quad \mu \geq 0_n, t_0^- \geq 0_m, t_0^+ \geq 0_s
 \end{aligned}$$

Making a second change of variables $t_{i0}^- = \beta s_{i0}^-$, $i = 1, \dots, m$, $t_{r0}^+ = \beta s_{r0}^+$, $r = 1, \dots, s$, $\mu_j = \beta \lambda_j$, $j = 1, \dots, n$, generates

$$\begin{aligned}
 \min_{\beta, \lambda, s_0^-, s_0^+} & \quad \beta \left(1 - \frac{1}{m} \sum_{i=1}^m \frac{s_{i0}^-}{x_{i0}} \right) \\
 \text{s.t.} & \quad \beta \left(1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{y_{r0}} \right) = 1 \\
 & \quad \beta \left(x_{i0} - \sum_{j=1}^n \lambda_j x_{ij} - s_{i0}^- \right) \geq 0, \quad \forall i \\
 & \quad -\beta \left(y_{r0} - \sum_{j=1}^n \lambda_j y_{rj} + s_{r0}^+ \right) \geq 0, \quad \forall r \\
 & \quad \beta \left(1 - \sum_{j=1}^n \lambda_j \right) = 0, \\
 & \quad \lambda \geq 0_n, s_0^- \geq 0_m, s_0^+ \geq 0_s
 \end{aligned}$$

The first restriction tells us two things. First that $\beta = \left(1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{y_{r0}} \right)^{-1} > 0$, which means that the objective function can be rewritten as shown below, and second, as a consequence, that all restrictions but the first can be simplified by deleting β . Therefore this nonlinear program can be rewritten as

$$\begin{aligned}
 \min_{\lambda, s_0^-, s_0^+} & \quad \left(1 - \frac{1}{m} \sum_{i=1}^m \frac{s_{i0}^-}{x_{i0}} \right) / \left(1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{y_{r0}} \right) \\
 \text{s.t.} & \quad x_{i0} - \sum_{j=1}^n \lambda_j x_{ij} - s_{i0}^- \geq 0, \quad \forall i \\
 & \quad y_{r0} - \sum_{j=1}^n \lambda_j y_{rj} + s_{r0}^+ \leq 0, \quad \forall r \\
 & \quad 1 - \sum_{j=1}^n \lambda_j = 0, \\
 & \quad \lambda \geq 0_n, s_0^- \geq 0_m, s_0^+ \geq 0_s
 \end{aligned}$$

The restrictions are exactly the restrictions of the additive program. The first two sets of restrictions can be equivalently written as equalities. Therefore, if we perform a third change of variables $\theta_i = 1 - \frac{s_{i0}^-}{x_{i0}}$, $i = 1, \dots, m$, $\phi_r = 1 + \frac{s_{r0}^+}{y_{r0}}$, $r = 1, \dots, s$, we finally get

$$\begin{aligned}
& \min_{\lambda, \theta, \phi} \quad \frac{1}{m} \sum_{i=1}^m \theta_i \Big/ \frac{1}{s} \sum_{r=1}^s \phi_r \\
& \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{i0}, \quad \forall i \\
& \quad \quad \sum_{j=1}^n \lambda_j y_{rj} \geq \phi_r y_{r0}, \quad \forall r \\
& \quad \quad \sum_{j=1}^n \lambda_j = 1 \\
& \quad \quad \lambda \geq 0_n, \theta \leq 1_m, \phi \geq 1_s
\end{aligned}$$

which is, exactly, the enhanced Russell graph program of Pastor et al. (1999), also known as the SBM (Slacks-Based Measure) (Tone (2001)).

Finally, considering all the above steps, we have at optimum

$$1 - L(x_0, y_0; LNC7) = \frac{1}{m} \sum_{i=1}^m \theta_i^* \Big/ \frac{1}{s} \sum_{r=1}^s \phi_r^* .$$

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