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A Dual Measure of Economies of Scope

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Abstract

Information on the economies of scope (or cost complementarities) between two or more output variables is traditionally obtained from the second derivative properties of an econometrically estimated multi-output cost function. However, in some instances the econometric estimation of a cost function may not be viable, because cost or input price data are not readily available or because the assumption of cost minimising behaviour is not appropriate in the industry at hand, perhaps due to government ownership or regulatory controls. In this paper we address this issue by utilising the duality between the cost and input distance functions in deriving an expression for a measure of economies of scope in terms of the derivatives of an input distance function. We derive expressions for the special cases of the CES and translog functional forms, and then provide an empirical illustration using sample data on Australian aged care facilities, an industry which is currently undergoing a major pricing and regulatory review. Our empirical results indicate that there is some evidence for existence of economies of scope between high care and low care patients, a result which is of particular interest to policy makers in this industry.

Key Words: Economies of Scope, Duality, Distance Function, Stochastic Frontier.

JEL Codes: D20, D24, L5

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1. Introduction

How does a firm decide if it should focus on the production of a single product or diversify into the production of two or more products? Many factors could be considered in such a decision. In most situations the two most important factors to consider are (i) the degree to which diversification could reduce risks associated with output price volatility, and (ii) the degree to which diversification (or specialisation) results in lower or higher unit costs. This latter factor has come to be known as *scope economies* (or diseconomies) in the economics literature, and is the focus of the current study.¹

Measures of economies of scope (or *cost complementarities*) are often derived from the parameters of an econometrically estimated multi-output cost function. For example, see the recent analysis of scale and scope economies in Chinese agricultural research institutes by Jin et al (2005). However, in some instances the econometric estimation of a cost function may not be viable, because cost or input price data are not readily available or because the assumption of cost minimising behaviour is not appropriate in the industry at hand.

This latter problem is particularly common in industries where businesses are regulated and/or government owned. Examples of such industries, where the assumption of cost minimising behaviour is unlikely to be widely applicable, include electricity supply,² telecommunications, railways, banking, education, health care and (arguably) publicly funded agricultural research institutes. However, it remains common practice for one to estimate dual cost functions in analysing scope economies in many of these industries, even though the duality assumptions that underpin these dual functions (for details see Färe and Primont 1995) are unlikely to be applicable in many cases. For recent examples of studies that use econometric estimates of cost functions to obtain measures of scope economies, see the analyses of: Japanese

¹ In their pioneering paper on this topic, Panzar and Willig (1981, p268) state that a multi-product firm is characterised by *economies of scope* when “it is less costly to combine two or more product lines in one firm than to produce them separately”.

² Examples of factors that may lead to distortions in cost minimising signals in the electricity industry include union pressure to increase employment, political pressure to “build more stuff”, and the possibility that a regulatory regime, such as rate of return regulation, may provide incentives for the firm to overcapitalise (see Averch and Johnston 1962).

electricity supply by Ida and Kuwahara (2004); Australian telecommunications by Bloch, Madden and Savage (2001); US railroads by Ivaldi and McCullough (2001); commercial banks in Singapore by Rezvanian and Mehdiian (2002); and US higher education by Laband and Lentz (2003).

In this paper we propose a method via which one can obtain measures of economies of scope without requiring estimates of the parameters of the cost function. We do this by first estimating a (multi-output) input distance function using econometric methods,³ and then exploiting the duality between the cost function and the input distance function to derive an expression for a measure of economies of scope (or cost complementarities) in terms of the derivatives of the input distance function. This approach has the advantages that the estimation of an input distance function does not require one to make questionable behavioural assumptions, such as cost minimising behaviour, nor does it require access to input price data, which is often difficult to obtain (especially in the case of capital inputs).

The methods described in this paper were motivated by an empirical study of Australian residential aged care facilities (nursing homes) that was commissioned by the Australian Government as part of a major pricing and regulatory review of this industry (see CEPA, 2003). One question that was of particular interest in this study was the degree to which aged care facilities benefited from economies of scope in providing services for both low care and high care patients (the latter being particularly frail). Traditionally, these two categories of patients were serviced by separate aged care facilities. However changes in Government policy have allowed facilities to serve both low care and high care patients, so as to permit what is commonly termed “aging in place”. This policy has the advantage that as a low care patient ages and becomes more frail, he or she is not then forced to move to another facility, which can be a traumatic experience. An empirical example that makes use of this aged care data is used to illustrate the methods that are proposed in this paper. Our results indicate that there is some evidence for existence of diseconomies of scope between high care and low care patients, a result that is likely to be of particular interest to policy makers in this industry.

³ The econometric estimation of input distance functions is a relatively recent advance. See, for example, Paul et al (2000), Atkinson and Primont (2002) and O’Donnell and Coelli (2005).

The remainder of this paper is organised into sections. In Section 2 we derive the basic duality relationship between the second order derivatives of cost and distance functions, which is then used in Section 3 to derive a formula to calculate economies of scope in terms of the derivatives of the distance function. In this section we also show how our (complex) mathematical expressions are simplified when the technology satisfies input homotheticity or constant returns to scale. In Section 4 we derive the economies of scope measures for two popular functional forms, the CES and translog. Section 5 is devoted to an empirical application where the proposed procedure is used in examining the presence of economies of scope in the residential aged care industry in Australia. Some concluding comments are then made in Section 6.

2. A Duality Relationship between Cost and Distance Functions

In this section we derive a relationship between the second derivatives of dual cost and input distance functions that permits the definition of an economies of scope measure in terms of an estimated input distance function. A number of previous papers have looked at derivative relationships between cost and distance functions in certain contexts. For example, see Hanoch (1975), Blackorby and Diewert (1979), Blackorby et al. (1981, 1989), Mundra and Russell (2002), and Atkinson and Primont (2002).⁴ However, this earlier work primarily addresses issues relating to input substitution and/or scale economies, but does not provide the results that are needed for an analysis of the scope economies issue.

We begin by defining the production technology set as

$$T = \{ \langle \mathbf{x}, \mathbf{y} \rangle : \mathbf{x} \text{ can produce } \mathbf{y} \},$$

where $\mathbf{x} \in \mathfrak{R}_+^n$ is a $n \times 1$ vector of input quantities and $\mathbf{y} \in \mathfrak{R}_+^m$ is a $m \times 1$ vector of output quantities. Under a fairly weak set of assumptions, one can equivalently

⁴ Deaton (1979) also does related work in relation to substitute and complementary goods in a consumer context.

represent this technology using the input distance function (see Färe and Primont 1995)

$$D(\mathbf{x}, \mathbf{y}) = \max \{ \lambda : \langle \mathbf{x} / \lambda, \mathbf{y} \rangle \in T \},$$

where λ is a scalar, such that $1 \leq \lambda < \infty$, and a value of $\lambda = 1$ implies that the firm is operating on the outer boundary of the production technology and hence is technically efficient (in the sense of Farrell 1957).

Using duality theory, the cost function may then be specified as a function of the input distance function (e.g. see Färe and Primont 1995)

$$C(\mathbf{p}, \mathbf{y}) = \min \{ \mathbf{p}'\mathbf{x} : D(\mathbf{x}, \mathbf{y}) \geq 1 \}, \quad (1)$$

where $\mathbf{p} \in \mathfrak{R}_+^n$ is a $n \times 1$ vector of input prices.

Given that the technology exhibits convexity and strong disposability in inputs, the cost function will be linearly homogenous, concave and non-decreasing with respect to \mathbf{p} and non-decreasing with respect to \mathbf{y} , and the input distance function will be linearly homogenous, concave and non-decreasing with respect to \mathbf{x} and non-decreasing with respect to \mathbf{y} (e.g., see Färe and Primont 1995).

The Lagrangian associated with the minimization problem (1) can be written as

$$L(\mathbf{x}, \lambda) = \mathbf{p}'\mathbf{x} - \lambda [D(\mathbf{x}, \mathbf{y}) - 1], \quad (2)$$

where the first order conditions are⁵

⁵ Throughout this paper, for any function denoted by $f(\mathbf{x}, \mathbf{y})$, $f_{\mathbf{x}}$ refers to the gradient of f with respect to vector \mathbf{x} , $f_{\mathbf{xx}}$ represents the Hessian matrix with respect to \mathbf{x} , and $f_{\mathbf{xy}}$ represents the matrix of second order derivatives of f with respect to vector \mathbf{x} and \mathbf{y} , respectively. Furthermore, $f_{\mathbf{y}}$, $f_{\mathbf{yy}}$ and $f_{\mathbf{yx}}$ are similarly defined.

$$\mathbf{p} - \lambda D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$$

and

$$D(\mathbf{x}, \mathbf{y}) - 1 = 0.$$

It has been shown (see Shephard 1970) that the optimal value of λ in the above relation is equal to $C(\mathbf{p}, \mathbf{y})$. Thus we have

$$\mathbf{p} - C(\mathbf{p}, \mathbf{y}) D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \tag{3}$$

and

$$D(\mathbf{x}, \mathbf{y}) - 1 = 0, \tag{4}$$

when the input quantity vector, \mathbf{x} , is that which minimises cost, for the given values of \mathbf{y} and \mathbf{p} , and the given technology.⁶

Applying the envelope theorem to (1) and (2) and the fact that $\lambda = C(\mathbf{p}, \mathbf{y})$, we obtain

$$C_{\mathbf{p}}(\mathbf{p}, \mathbf{y}) = \mathbf{x} \tag{5}$$

and

$$C_{\mathbf{y}}(\mathbf{p}, \mathbf{y}) = -D_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) C(\mathbf{p}, \mathbf{y}). \tag{6}$$

The relation in (5) is the well known *Shephard's Lemma*. In the following theorem we use relations (3)-(6) to obtain an expression for the second order derivatives of the cost function in terms of the derivatives of the distance function.

Theorem 1: Given that the cost function and its dual distance function are twice continuously differentiable, then at any point (\mathbf{p}, \mathbf{y})

⁶ It is important to note that the scope measures that we introduce later in this paper do not hinge on an assumption of cost minimising behaviour, even though duality results are used in their derivation. The key to this apparent contradiction is the use of the concept of shadow cost minimisation, which we discuss shortly.

$$\begin{bmatrix} C_{py} \\ \frac{1}{C}C_{yy} \end{bmatrix} = \begin{bmatrix} -[D_{xx} + D_x D_x']^{-1} D_{xy} \\ D_y D_y' - D_{yy} + D_{yx} [D_{xx} + D_x D_x']^{-1} D_{xy} \end{bmatrix}, \quad (7)$$

where $C_{py} = \left\{ \frac{\partial^2 C(\mathbf{p}, \mathbf{y})}{\partial \mathbf{p} \partial \mathbf{y}} \right\}$, $D_{yx} = \left\{ \frac{\partial^2 D(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y} \partial \mathbf{x}} \right\}$ and $\mathbf{x} = C_{\mathbf{p}}(\mathbf{p}, \mathbf{y})$. C_{yy} , D_{yy} and D_{xx} are defined similarly.

Proof: Our proof is based on relations (3), (4) and (6) discussed above. Namely,

$$D(\mathbf{x}, \mathbf{y}) = 1,$$

$$C(\mathbf{p}, \mathbf{y}) D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) = \mathbf{p},$$

and

$$C_{\mathbf{y}}(\mathbf{p}, \mathbf{y}) = -D_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) C(\mathbf{p}, \mathbf{y}).$$

Taking partial derivatives with respect to \mathbf{y} in each of these three relations, and using the fact that $\mathbf{x} = C_{\mathbf{p}}(\mathbf{p}, \mathbf{y})$, we obtain

$$D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) C_{py}(\mathbf{p}, \mathbf{y}) + D_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) = 0,$$

$$C(\mathbf{p}, \mathbf{y}) \left[D_{xx}(\mathbf{x}, \mathbf{y}) C_{py}(\mathbf{p}, \mathbf{y}) + D_{xy}(\mathbf{x}, \mathbf{y}) \right] + D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) C_{\mathbf{y}}(\mathbf{p}, \mathbf{y})' = 0$$

and

$$C_{yy}(\mathbf{p}, \mathbf{y}) = -D_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) C_{\mathbf{y}}(\mathbf{p}, \mathbf{y})' - C(\mathbf{p}, \mathbf{y}) \left[D_{yx}(\mathbf{x}, \mathbf{y}) C_{py}(\mathbf{p}, \mathbf{y}) + D_{yy}(\mathbf{x}, \mathbf{y}) \right].$$

Rearranging this in matrix form provides

$$\begin{bmatrix} 0 & C(\mathbf{p}, \mathbf{y}) D_{\mathbf{x}}(\mathbf{x}, \mathbf{y})' & \mathbf{0} \\ D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) & C(\mathbf{p}, \mathbf{y}) D_{xx}(\mathbf{x}, \mathbf{y}) & \mathbf{0} \\ D_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) & C(\mathbf{p}, \mathbf{y}) D_{yx}(\mathbf{x}, \mathbf{y}) & \mathbf{I} \end{bmatrix} \begin{bmatrix} C_{\mathbf{y}}(\mathbf{p}, \mathbf{y})' \\ C_{py}(\mathbf{p}, \mathbf{y}) \\ C_{yy}(\mathbf{p}, \mathbf{y}) \end{bmatrix} = \begin{bmatrix} -C(\mathbf{p}, \mathbf{y}) D_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) \\ -C(\mathbf{p}, \mathbf{y}) D_{xy}(\mathbf{x}, \mathbf{y}) \\ -C(\mathbf{p}, \mathbf{y}) D_{yy}(\mathbf{x}, \mathbf{y}) \end{bmatrix}. \quad (8)$$

Now, to obtain the values of interest (the second matrix on the left hand side of equation 8) we need to invert the first matrix on left hand side of this equation.

First, let $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ [D_{\mathbf{y}} & CD_{yx}] & \mathbf{I} \end{bmatrix}^{-1}$, where $\mathbf{A} = \begin{bmatrix} 0 & CD_{\mathbf{x}}' \\ D_{\mathbf{x}} & CD_{xx} \end{bmatrix}$ and \mathbf{I} is the identity

matrix, where the arguments on the cost and distance functions have been suppressed

to avoid notational clutter. Then using the standard formula for the inversion of partitioned matrices we obtain

$$\mathbf{B}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -[D_y \quad CD_{yx}]\mathbf{A}^{-1} & \mathbf{I} \end{bmatrix}^{-1}.$$

Unfortunately, we cannot use the standard inversion formula to obtain \mathbf{A}^{-1} because both diagonal elements in \mathbf{A} are non-invertible. However, it can be shown (see Appendix 1) that

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{\mathbf{x}'}{D} \\ \frac{C^{-1}\mathbf{x}}{D} & C^{-1}[D_{xx} + D_x D_x']^{-1} - C^{-1}\frac{\mathbf{x}\mathbf{x}'}{D^2} \end{bmatrix}.$$

Using this expression for \mathbf{A}^{-1} , we then obtain \mathbf{B}^{-1} as

$$\mathbf{B}^{-1} = \begin{bmatrix} 0 & \frac{\mathbf{x}'}{D} & \mathbf{0} \\ \frac{C^{-1}\mathbf{x}}{D} & C^{-1}[D_{xx} + D_x D_x']^{-1} - C^{-1}\frac{\mathbf{x}\mathbf{x}'}{D^2} & \mathbf{0} \\ \frac{-D_y}{D} & -CD_{yx} \left\{ C^{-1}[D_{xx} + D_x D_x']^{-1} - C^{-1}\frac{\mathbf{x}\mathbf{x}'}{D^2} \right\} - \frac{D_y \mathbf{x}'}{D} & \mathbf{I} \end{bmatrix}.$$

Pre-multiplication of both sides of (8) by \mathbf{B}^{-1} provides

$$\begin{bmatrix} C_y' \\ C_{py} \\ C_{yy} \end{bmatrix} = \begin{bmatrix} -CD_y' \\ -\frac{\mathbf{x}D_y'}{D} - [D_{xx} + D_x D_x']^{-1} D_{xy} + \frac{\mathbf{x}D_y'}{D^2} \\ C \left\{ \frac{2D_y D_y}{D} - \frac{D_y D_y'}{D^2} + D_{yx} [D_{xx} + D_x D_x']^{-1} D_{xy} - D_{yy} \right\} \end{bmatrix}.$$

Then, since $D(\mathbf{x}, \mathbf{y}) = 1$ we obtain

$$\begin{bmatrix} C_{py} \\ \frac{1}{C} C_{yy} \end{bmatrix} = \begin{bmatrix} -[D_{xx} + D_x D_x']^{-1} D_{xy} \\ D_y D_y' - D_{yy} + D_{yx} [D_{xx} + D_x D_x']^{-1} D_{xy} \end{bmatrix}.$$

Q.E.D.

3. Economies of Scope

According to Panzar and Willig (1981), there are economies (diseconomies) of scope where it is less (more) costly to produce two or more outputs jointly within one firm than to produce them separately. For ease of presentation we consider the two commodity case where a firm produces two goods and output levels are denoted by y_1 and y_2 . In formal terms, it is said that there are (weak) economies of scope between outputs 1 and 2 if⁷

$$C(y_1, y_2; \mathbf{p}) \leq C(y_1, 0; \mathbf{p}) + C(0, y_2; \mathbf{p}). \quad (9)$$

Baumol, Panzar and Willig (1988, p.72-73) provide a more general definition appropriate in the n commodity case. One can alternatively state that there are diseconomies of scope between outputs 1 and 2 if the inequality above is reversed indicating that it is more costly to produce outputs 1 and 2 jointly than to produce them individually by two different firms.

This definition of economies of scope cannot be verified directly when the cost function assumes forms in which outputs appear in a logarithmic or a reciprocal form. In such cases the following sufficient condition can be used in checking for the presence of economies of scope. It has been also shown that a sufficient condition for the presence of economies of scope⁸ to hold is that the cost function exhibits weak cost complementarities, i.e.,

⁷ There are *strong* economies of scope if a strict inequality holds in equation (9).

⁸ One should assume that we refer to *weak* economies of scope unless otherwise specified.

$$\frac{\partial^2 C(\tilde{\mathbf{y}}, \mathbf{p})}{\partial y_i \partial y_j} \leq 0, \quad i \neq j, \quad (10)$$

for all $\tilde{\mathbf{y}}$, such that $\mathbf{0} \leq \tilde{\mathbf{y}} \leq \mathbf{y}$ (Baumol, Panzar and Willig 1988). It can also be shown that a sufficient condition that for the presence of diseconomies of scope is that

$$\frac{\partial^2 C(\tilde{\mathbf{y}}, \mathbf{p})}{\partial y_i \partial y_j} \geq 0, \quad i \neq j, \quad (11)$$

for all $\tilde{\mathbf{y}}$, such that $\mathbf{0} \leq \tilde{\mathbf{y}} \leq \mathbf{y}$. Knowledge of the presence of diseconomies of scope can be just as important as the presence of economies of scope. See Baumol, Panzar and Willig (1988) for discussion of some of the implications of the presence of economies (or diseconomies) of scope for the organisational structure of the industry under consideration.

Our focus in this paper is on the sufficient conditions in equations (10) and (11). One could argue for the use of this derivative based measure for various reasons. First, it can be applied to functional forms which are not defined for zero values of output, while the measure in equation (9) cannot. Second, it avoids the danger of extrapolating the estimated cost function out into parts of the data space in which one may have little or no data at all. A third reason, specific to this paper, is that the derivative relationships derived in the previous section allow us to calculate this scope economies measure from the parameters of an estimated distance function. This means that we do not need to estimate the cost function itself. This can be advantageous in situations where data on costs or input prices are not readily available, or when the assumption of cost minimisation is unlikely to be valid.

Utilising Theorem 1, the economies of scope between outputs i and j can be expressed in terms of the derivatives of the input distance function as

$$\frac{1}{C} \frac{\partial^2 C}{\partial y_i \partial y_j} = \frac{\partial D}{\partial y_i} \frac{\partial D}{\partial y_j} - \frac{\partial^2 D}{\partial y_i \partial y_j} + \left(\frac{\partial^2 D}{\partial y_i \partial x_1} \quad \dots \quad \frac{\partial^2 D}{\partial y_i \partial x_n} \right) \begin{pmatrix} \frac{\partial^2 D}{\partial x_1^2} + \frac{\partial D}{\partial x_1} \frac{\partial D}{\partial x_1} & \dots & \frac{\partial^2 D}{\partial x_1 \partial x_n} + \frac{\partial D}{\partial x_1} \frac{\partial D}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 D}{\partial x_n \partial x_1} + \frac{\partial D}{\partial x_n} \frac{\partial D}{\partial x_1} & \dots & \frac{\partial^2 D}{\partial x_n^2} + \frac{\partial D}{\partial x_n} \frac{\partial D}{\partial x_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial^2 D}{\partial x_1 \partial y_j} \\ \vdots \\ \frac{\partial^2 D}{\partial x_n \partial y_j} \end{pmatrix}. \quad (12)$$

Weak economies of scope exist if the right hand side of equation (12) is non-positive.

Shadow cost minimisation

As noted in section 2, the duality between the cost function and the input distance function relies upon an assumption of cost minimising behaviour. Thus it follows that the above economies of scope measure also relies on this assumption. This observation appears to “throw cold water” on one of the motivations for the use of this newly proposed measure – which is to allow one to avoid the need to estimate a cost function in those industries where cost minimising behaviour is unlikely to be a valid assumption.

This apparent problem (or inconsistency) can be solved by making the weaker assumption that the firms are *shadow cost minimisers*. This involves the assumption that the firm is seeking to minimise costs relative to a vector of *shadow input prices* (which are not observed by the econometrician).⁹ In this case the cost function in equation (1) is converted into a shadow cost function

$$C(\mathbf{p}^*, \mathbf{y}) = \min \{ \mathbf{p}^* \mathbf{x} : D(\mathbf{x}, \mathbf{y}) \geq 1 \},$$

where $\mathbf{p}^* \in \mathfrak{R}_+^n$ is a $n \times 1$ vector of input shadow prices, and the derivations in the preceding sections follow in the same manner.¹⁰

Thus, in empirical implementations of this scope economies measure, when we evaluate the distance function derivatives at an observed data point (\mathbf{x}, \mathbf{y}) , and then use these to calculate the cost function derivatives, we are effectively evaluating the derivatives of the (unobserved) cost function at the data point $(\mathbf{p}^*, \mathbf{y})$, where \mathbf{p}^* is a vector of input shadow prices. Furthermore, in the event that the firm is actually minimising *observed* cost, \mathbf{p} and \mathbf{p}^* will coincide, and hence the scope economies

⁹ An often quoted example of a situation in which shadow prices can differ from observed (or market) prices is the case where rate of return regulation affects the price of capital that is perceived by a regulated firm. See Averch and Johnston (1962) for further discussion.

¹⁰ This notion of shadow cost minimisation is regularly utilised in empirical analyses of regulated industries. For example, see Balk (1998).

measure obtained will correspond to that which would have been obtained directly from the cost function (abstracting from possible differences arising during econometric estimation).

Some special cases

Given that the expression in equation (9) is fairly messy, one question of interest is: Can anything conclusive be said about the nature of economies of scope between outputs i and j by simply looking at the corresponding second order cross partial derivatives of the input distance function? One can state that if this derivative is positive (negative) then there is some likelihood that scope economies (diseconomies) exist. But, this information alone is normally not enough for one to be able make a more definite statement.

However, if the technology satisfies certain restrictions, such as input homotheticity, this situation can change. A technology is said to exhibit input homotheticity if the associated distance function can be written as the product of two functions, such that $D(\mathbf{y}, \mathbf{x}) = g(\mathbf{x})h(\mathbf{y})$ (Balk 1998, pp 16).

Theorem 2: Suppose that the technology exhibits input homotheticity, then

$$\frac{1}{C} C_{yy} = 2D_y D_y' - D_{yy}. \quad (10)$$

Proof: It is easy to show that the input homotheticity implies

$$D_{xy} = \frac{D_x D_y'}{D}.$$

Using this result, and noting that $D = 1$, equation (9) becomes¹¹

$$\frac{1}{C} C_{yy} = D_y D_y' - D_{yy} + D_y D_x' (D_{xx} + D_x D_x')^{-1} D_x D_y'.$$

¹¹ The result that $D=1$ follows from the fact that we are talking about derivative properties on the surface of the production technology.

Since D is homogenous of degree one in \mathbf{x} we can write

$$D_{\mathbf{x}}' (D_{\mathbf{xx}} + D_{\mathbf{x}} D_{\mathbf{x}}')^{-1} D_{\mathbf{x}} = 1,$$

and hence

$$\frac{1}{C} C_{\mathbf{yy}} = 2D_{\mathbf{y}} D_{\mathbf{y}}' - D_{\mathbf{yy}}.$$

Q.E.D.

As we see from the above theorem, under input homotheticity, equation (9) simplifies to

$$\frac{1}{C} \frac{\partial^2 C}{\partial y_i \partial y_j} = 2 \frac{\partial D}{\partial y_i} \frac{\partial D}{\partial y_j} - \frac{\partial^2 D}{\partial y_i \partial y_j}. \quad (11)$$

In this case we can say that a positive sign for a mixed second order derivative of distance function with respect to outputs is a necessary condition for the existence of economies of scope but it is not sufficient, while a negative sign is both a necessary and sufficient condition for existence of diseconomies of scope.¹²

By further restricting the technology, we can obtain another useful relation, this time in terms of the output distance function.

Theorem 3: Suppose that the technology satisfies global constant returns to scale, then

$$\frac{1}{C} C_{\mathbf{yy}}(\mathbf{p}, \mathbf{y}) = D_{\mathbf{yy}}^o(\mathbf{x}, \mathbf{y}),$$

where $D^o(\mathbf{x}, \mathbf{y})$ is the output distance function.¹³

¹² In an econometric analysis of the production of small holder farmers in Papua New Guinea, Coelli and Fleming (2004) estimate an (input homothetic) restricted translog output distance function and then use the second cross partial derivatives of the estimated output distance function (with respect to outputs i and j) to investigate the existence of economies of scope. They emphasise that this measure is not equivalent to the traditional scope economies measure derived from a cost function, and hence coin the term “economies of diversification” for this measure, to emphasis this distinction.

¹³ The output distance function is an alternative way of characterising a production technology. It is defined as $D_o(\mathbf{x}, \mathbf{y}) = \min\{\theta: \langle \mathbf{x}, \mathbf{y}/\theta \rangle \in T\}$. For further details see Färe and Primont (1995).

Proof: Under global constant returns to scale, at any point (\mathbf{x}, \mathbf{y}) , we can write (Färe and Primont 1995, pp 24)

$$D^o(\mathbf{x}, \mathbf{y}) = \frac{1}{D(\mathbf{x}, \mathbf{y})}.$$

Taking second derivatives with respect to \mathbf{y} we obtain

$$D_{yy}^o = \frac{2D_y D_y' D - D_{yy} D^2}{D^3}.$$

Plus, given that $D^o(\mathbf{x}, \mathbf{y}) = 1$ we have

$$D_{yy}^o = 2D_y D_y' - D_{yy}. \quad (12)$$

Since a technology that exhibits global constant returns to scale is necessarily input homothetic, from theorem (2) we have

$$\frac{1}{C} C_{yy} = 2D_y D_y' - D_{yy}. \quad (13)$$

Then from (12) and (13) we can see that

$$\frac{1}{C} C_{yy}(\mathbf{p}, \mathbf{y}) = D_{yy}^o(\mathbf{x}, \mathbf{y}). \quad (14)$$

Q.E.D.

Thus, under global constant returns to scale, a negative (positive) sign for the cross derivative of an output distance function is both a necessary and sufficient condition for the existence of economies (diseconomies) of scope.

4. Some Examples

In this section we derive measures of economies of scope associated with the CES and translog input distance functions.

A CES Input Distance Function

A CES input distance function may be defined as

$$D = B(\mathbf{x})^{1/\rho} A(\mathbf{y})^{-1/\delta},$$

where $B(\mathbf{x}) = \left[\sum_{i=1}^n \alpha_i x_i^\rho \right]$, $A(\mathbf{y}) = \left[\sum_{j=1}^m \beta_j y_j^\delta \right]$ and $\alpha_i > 0$, $\beta_j > 0$, $-\infty < \rho \leq 1$, $\delta \geq 1$.

We first derive

$$2 \frac{\partial D}{\partial y_i} \frac{\partial D}{\partial y_j} = 2 \beta_i \beta_j y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{\frac{-2-2\delta}{\delta}} B(\mathbf{x})^{\frac{2}{\rho}}.$$

We can then use the fact $D(\mathbf{y}, \mathbf{x}) = 1$ implies $B(\mathbf{x})^{1/\rho} A(\mathbf{y})^{-1/\delta} = 1$ to obtain

$$2 \frac{\partial D}{\partial y_i} \frac{\partial D}{\partial y_j} = 2 \beta_i \beta_j y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{-2}$$

and

$$\frac{\partial^2 D}{\partial y_i \partial y_j} = \beta_i \beta_j (-1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{\frac{-1-2\delta}{\delta}} B(\mathbf{x})^{\frac{1}{\rho}} = \beta_i \beta_j (-1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{-2}.$$

Since the CES distance function satisfies input homotheticity, the economies of scope measure will be equal to

$$2 \frac{\partial D}{\partial y_i} \frac{\partial D}{\partial y_j} - \frac{\partial^2 D}{\partial y_i \partial y_j} = \beta_i \beta_j (1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{-2}.$$

We can check the validity of the above derivation by using the cost function. It can be shown that the dual cost function is of the following form

$$C = A(\mathbf{y})^{\frac{1}{\delta}} \mathbf{a}^{\frac{-1}{\rho}} \left[\sum_{i=1}^n \alpha_i^{\frac{1}{\rho-1}} p_i^{\frac{\rho}{\rho-1}} \right]^{\frac{\rho-1}{\rho}},$$

We calculate

$$\frac{\partial^2 C}{\partial y_i \partial y_j} = \beta_i \beta_j (1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{\frac{1-2\delta}{\delta}} \mathbf{a}^{\frac{-1}{\rho}} \left[\sum_{i=1}^n \alpha_i^{\frac{1}{\rho-1}} p_i^{\frac{\rho}{\rho-1}} \right]^{\frac{\rho-1}{\rho}},$$

which provides

$$\frac{1}{C} \frac{\partial^2 C}{\partial y_i \partial y_j} = \beta_i \beta_j (1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{-2}.$$

Thus, both approaches result in the same expression.

The CES also satisfies global constant returns to scale, so the results of the theorem (3) can be applied to it. It can be shown that the dual output distance function has the following form (Färe and Primont 1995):

$$D^o = B(\mathbf{x})^{-1/\rho} A(\mathbf{y})^{1/\delta}.$$

Taking second derivatives (with respect to y_i and y_j), and using the fact that $D^o(\mathbf{y}, \mathbf{x}) = 1$ implies $B(\mathbf{x})^{-1/\rho} A(\mathbf{y})^{1/\delta} = 1$, we obtain

$$\frac{\partial^2 D}{\partial y_i \partial y_j} = \beta_i \beta_j (1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{\frac{1-2\delta}{\delta}} B(\mathbf{x})^{-1/\rho} = \beta_i \beta_j (1-\delta) y_i^{\delta-1} y_j^{\delta-1} A(\mathbf{y})^{-2}.$$

As expected, this is equal to the standard economies of scope measure derived from the cost function.

A Translog Input Distance Function

A translog input distance function may be defined as (Paul *et al* 2000)

$$\begin{aligned} \ln D(\mathbf{x}, \mathbf{y}) = & \alpha_0 + \sum_{i=1}^n \alpha_i \ln(x_i) + \sum_{i=1}^m \beta_i \ln(y_i) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} \ln(x_i) \ln(x_j) \\ & + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \beta_{ij} \ln(y_i) \ln(y_j) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \gamma_{ij} \ln(x_i) \ln(y_j). \end{aligned} \quad (15)$$

To be a valid input distance function, the specification must satisfy the following homogeneity restrictions

$$\sum_{i=1}^n \alpha_i = 1, \quad \sum_{k=1}^n \alpha_{ik} = 0, \quad i=1,2,\dots,n, \quad \text{and} \quad \sum_{j=1}^m \gamma_{ij} = 0, \quad i=1,2,\dots,n.$$

In addition to these restrictions, it must satisfy the following symmetry restrictions due to Young's theorem:

$$\alpha_{ik} = \alpha_{ki} \quad \text{and} \quad \beta_{jl} = \beta_{lj}, \quad \text{for all } i, k, j \text{ and } l.$$

In the case of the translog functional form, the calculation of economies of scope measures using equation (9) requires calculation of various derivatives: \mathbf{D}_x , \mathbf{D}_y , \mathbf{D}_{xx} , \mathbf{D}_{yy} and \mathbf{D}_{yx} . At first glance, this might seem rather difficult and tedious and one might alternatively think of using numerical derivatives. However, we now show that the analytical derivatives can be obtained fairly easily for the translog function. First we define

$$g_k = \frac{\ln D(\mathbf{x}, \mathbf{y})}{\partial \ln x_k} = \alpha_k + \sum_{j=1}^n \alpha_{kj} \ln x_j + \sum_{j=1}^m \gamma_{kj} \ln y_j$$

and

$$h_l = \frac{\ln D(\mathbf{x}, \mathbf{y})}{\partial \ln y_l} = \beta_l + \sum_{j=1}^m \beta_{lj} \ln y_j + \sum_{i=1}^n \gamma_{il} \ln x_i.$$

One can then simply obtain the first order derivatives as

$$\mathbf{D}_x = \{d_k^x\}, \text{ where } d_k^x = \frac{D}{x_k} g_k,$$

and

$$\mathbf{D}_y = \{d_l^y\}, \text{ where } d_l^y = \frac{D}{y_l} h_l.$$

To obtain the matrix of second order derivatives we need to take derivatives from the above relations. After some manipulation we obtain the following results

$$\mathbf{D}_{xx} = \{d_{kl}^{xx}\}, \text{ where } d_{kl}^{xx} = \begin{cases} \frac{D}{x_k^2} (g_k^2 - g_k + \alpha_{kk}), & k = l \\ \frac{D}{x_k x_l} (g_k g_l + \alpha_{kl}), & k \neq l \end{cases},$$

$$\mathbf{D}_{yy} = \{d_{kl}^{yy}\}, \text{ where } d_{kl}^{yy} = \begin{cases} \frac{D}{y_k^2} (h_k^2 - h_k + \beta_{kk}), & k = l \\ \frac{D}{y_k y_l} (h_k h_l + \beta_{kl}), & k \neq l \end{cases}$$

and

$$\mathbf{D}_{yx} = \{d_{lk}^{yx}\}, \text{ where } d_{lk}^{yx} = \frac{D}{y_l x_k} (h_l g_k + \gamma_{lk}).$$

Even simpler expressions for these derivatives can be obtained if one is interested in evaluating them at the means of the sample data. In this instance, if one mean-corrects the sample data (i.e., each variable is deflated by its mean) prior to estimating the distance function, the first derivatives (evaluated at the sample means) will simply be equal to the first order coefficients. Thus, the matrix of scope measures (see Appendix 2) becomes

$$\begin{bmatrix} \beta_1 - \beta_{11} & \cdots & -\beta_{1m} \\ \vdots & \ddots & \vdots \\ -\beta_{m1} & \cdots & \beta_m - \beta_{mm} \end{bmatrix} + \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_n \beta_1 + \gamma_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_1 \beta_m + \gamma_{1m} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \begin{bmatrix} 2\alpha_1^2 + \alpha_{11} - \alpha_1 & \cdots & 2\alpha_1 \alpha_n + \alpha_{1n} \\ \vdots & \ddots & \vdots \\ 2\alpha_n \alpha_1 + \alpha_{n1} & \cdots & 2\alpha_n^2 + \alpha_{nn} - \alpha_n \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_1 \beta_m + \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_n \beta_1 + \gamma_{n1} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \quad (16)$$

The translog formulas derived in this section are utilised in the empirical application in the next section.

5. Empirical Application to the Australian Residential Aged Care Industry

In this section, we use an empirical application to demonstrate how the results derived in previous sections can be used to investigate the presence of economies of scope using the derivatives of an input distance function. The data for this empirical application are taken from a recent study of the efficiency of residential aged care facilities in Australia undertaken by CEPA (2003), and involves survey data on the activities of 421 aged care facilities in the 2001/02 financial year.

This study of the efficiency of residential aged care facilities in Australia was commissioned by the Australian Government as part of a major pricing and regulatory review of this industry (see CEPA, 2003). One question that is of some interest is the degree to which aged care facilities benefit from economies of scope in providing services for both low care and high care patients (the latter being particularly frail). Not long ago, these two categories of patients had been traditionally serviced by separate aged care facilities in Australia. However changes in Government policy allowed facilities to serve both low care and high care patients, so as to permit what is commonly termed “aging in place”. This policy has the advantage that as a low care

patient ages and becomes more frail, he or she is not then forced to move to another facility so as to receive a higher level of care, which can be an distressing experience. Furthermore, the policy also has the advantage that a married couple can remain in the one ACF, even when their degrees of frailty differ significantly.

In this application, an input distance function was selected (instead of a cost function) in this industry for two reasons. First, we felt that an assumption of cost minimising behaviour was not reasonable in this industry, where the majority of aged care facilities are managed on a not-for-profit basis by church organisations, governments and community groups. Second, reliable input price data (for example on the wages of nurses across different geographical areas) was not readily available, nor was it likely to differ significantly over geographical regions, if such data was accessible. Hence, the input distance function was the logical choice in this instance.

The input distance function model that is estimated in this study has two output variables: high care resident services (y_1) and low care resident services (y_2), and two input variables: variable inputs (x_1) and capital inputs (x_2). The two output variables are in fact weighted bed day measures, obtained by aggregating a number of different resident categories that are distinguished by the government in their funding model. In providing funding support to aged care facilities, the Commonwealth Government of Australia identify eight different categories under which residential care services are provided, according to the degree of services required by each resident. Four of these categories fall into a group known as high care services and four into another group known as low care services. The weights used in constructing our two weighted bed day output variables are based upon the funding formula used by the government (see CEPA, 2003, p. 33).

The *variable inputs* variable is calculated as the total variable costs of the aged care facility (which includes labour costs, costs of materials and the costs of other variable inputs, such as outsourced services). Initially we planned to provide separate labour and non-labour variables, however given the degree of outsourcing of services (especially in catering and cleaning) we decided that such a distinction would be

artificial and hence have aggregated these items together.¹⁴ For capital, the reported capital measures were found to be unsatisfactory because they did not correspond to an economic notion of capital, being based upon historical cost measures that have been depreciated using a variety of different depreciation schedules. Consequently, the number of beds is used as a proxy of the capital input, in line with many other studies in the health care sector.¹⁵

The translog input distance function is estimated using Bayesian stochastic frontier methods in this study. The Bayesian method is chosen for two reasons. First, because it allows one to easily impose regularity conditions implied by economic theory. Second, because it allows us to easily compute the standard errors for the scope economies measure, which otherwise would be a complicated exercise, given the fact that it is a complex non-linear function of the estimated parameters.

To simplify notation, we denote the translog input distance function from equation (15) as

$$\ln D_i(\mathbf{y}_i, \mathbf{x}_i) = tl(\ln y_{1i}, \ln y_{2i}, \ln x_{1i}, \ln x_{2i}) + v_i, \quad (17)$$

where v_i is a random disturbance term capturing measurement and specification errors. Exploiting the homogeneity of input distance function with respect to input quantities and Euler's theorem, we obtain

$$-\ln x_{1i} + \ln D_i = tl(\ln y_{1i}, \ln y_{2i}, \ln x_{2i} - \ln x_{1i}, \ln x_{1i} - \ln x_{1i}) + v_i,$$

or equivalently

$$-\ln x_{1i} = tl(\ln y_{1i}, \ln y_{2i}, \ln x_{2i} - \ln x_{1i}) - \ln D_i + v_i. \quad (18)$$

¹⁴ As a check, we also ran the model with separate labour and non-labour variable input variables. In this model the latter variable was found to be statistically insignificant and furthermore our scope economies measure changed only minimally.

¹⁵ For example, see Farsi and Filippini (2004).

Given that D_i (which varies between 1 and infinity) represents the distance that the observation lies below the (stochastic) production technology, $\ln D_i (= u_i)$ will take a value between 0 and infinity.

In order to allow us to identify these two error components in cross-sectional data, we make the standard distributional assumptions that are generally made in stochastic frontier analysis models. That is, we assume that v_i is *iid* $N(0, \sigma_v^2)$ and distributed independently of u_i which is *iid* $N^+(0, \sigma_u^2)$, a half normal random variable reflecting the one-sided nature of this distance measure.

The model is estimated using the Markov chain Monte Carlo (MCMC) routine in WinBUGS,¹⁶ and makes use of the stochastic frontier analysis code described in Griffin and Steel (2005). We place non-informative priors on all parameters in the model, with the exception of the two variance parameters, for which we use Gamma(0.001, 0.001) priors for their inverses, which are diffuse priors that ensure non-negativity. Estimation involves 25,000 iterations, with a burn-in period of 10,000 iterations. Posterior means and standard deviations of the parameters of the input distance function based upon the remaining 15,000 iterations are reported in Table 1.

The estimates reported in Table 1 all have standard deviations that are small relative to their means, indicating that the model is well estimated.¹⁷ The signs and magnitudes of the first-order coefficients are generally as expected, with the α_i being positive (i.e., extra input increasing the distance to the frontier, *ceteris paribus*) and the β_i being negative (i.e., extra output decreasing the distance to the frontier, *ceteris paribus*).¹⁸ The returns to scale elasticity, evaluated at the sample means, is $-1/(\beta_1 + \beta_2) = 1.07$, providing evidence of mildly increasing returns to scale.¹⁹

¹⁶ See <http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/contents.shtml>.

¹⁷ The homogeneity and symmetry restrictions can be used to obtain estimates of those parameters not reported in Table 1.

¹⁸ The estimated function was also found to satisfy the required monotonicity and curvature conditions, at the sample means.

¹⁹ The model was also estimated using maximum likelihood methods using the stochastic frontier software of Coelli (1996). The results obtained were very similar to these Bayesian estimates. This is not surprising given the use of non-informative and diffuse priors.

Table 1: Bayesian MCMC estimates

	Mean	St. Dev.
α_0	0.3459	0.0104
α_1	0.6573	0.0258
α_{11}	0.3565	0.0346
β_1	-0.5870	0.0110
β_2	-0.3468	0.0091
β_{11}	-0.1514	0.0101
β_{12}	0.1449	0.0102
β_{22}	-0.1368	0.0108
γ_{11}	0.1122	0.0271
γ_{12}	-0.1568	0.0292
σ_v^2	0.0083	0.0019
σ_u^2	0.0292	0.0065

The measure economies of scope between high care and low care residents (evaluated at the sample means) was calculated to be -0.079 with a standard deviation of 0.035. This suggests the existence of significant economies of scope in delivering high care and low care services in residential aged care facilities in Australia.²⁰ We hypothesise that this result may be a consequence of the fact that low care facilities need to have a minimum amount of trained nursing staff and medical equipment (e.g., monitors, oxygen supply) on hand to deal with possible emergencies, and hence these inputs are likely to be underutilised to some extent. However, when an ACF has both high care and low care patients, the low care section can perhaps draw upon these types of inputs from the high care facility when they are needed.

Irrespective of the exact reasons for these scope economies, these results suggest that the introduction of the aging in place policy has not only had benefits in terms of

²⁰ Measures of scope economies for each observation in the data set are also available from the authors on request.

reducing the amount of emotional trauma associated with forced movements between ACFs, but also may have positive benefits in terms of cost reductions associated with scope economies. However, given the data problems that we have encountered in this application, further research is required before this latter point can be confirmed with confidence.

6. Conclusions

In this paper we provide a framework for obtaining measures of economies of scope from an estimated input distance function when one is unable to obtain (reliable) estimates of the cost function parameters. We note that economies of scope measures are likely to be of particular policy interest in those industries where a non-profit ownership structure and/or regulatory interventions are likely to cause one to question the applicability of the cost minimisation assumption that underlies the estimation of dual cost functions.

We exploit duality theory to obtain analytical expressions for the relationship between the second-order derivatives of the cost function and the derivatives of the distance function, under an assumption of shadow cost minimising behaviour. We also provide a number of simpler expressions for those situations when the underlying technology satisfies certain restrictions, such as input homotheticity and constant returns to scale. The feasibility and applicability of the analytical results are demonstrated using data on the Australian residential aged care facilities. We estimate a translog stochastic input distance function for this industry, finding evidence of economies of scope in the provision of high and low care services. This result is of particular policy interest to the Australian aged care industry, where a significant pricing and regulatory review has been undertaken in recent years.

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Appendix 1: Derivation of \mathbf{A}^{-1}

Using Proposition 2.31a in Dhrymes (2000, pp 46), and the fact that D is homogenous in \mathbf{x} , we obtain

$$\begin{bmatrix} 0 & CD'_x \\ D_x & CD_{xx} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{\mathbf{x}'}{D} \\ \frac{C^{-1}\mathbf{x}}{D} & C^{-1}[D_{xx} + D_x D'_x]^{-1} - C^{-1} \frac{\mathbf{xx}'}{D^2} \end{bmatrix}.$$

This result can be verified via direct multiplication. Using the following relations that follow from homogeneity of D with respect to \mathbf{x} :

$$D_x \mathbf{x} = D,$$

$$[D_{xx} + D_x D'_x] \left(\frac{\mathbf{x}}{D} \right) = D_x \Rightarrow \left(\frac{\mathbf{x}}{D} \right) = [D_{xx} + D_x D'_x]^{-1} D_x,$$

and

$$\begin{aligned} \left[\mathbf{I} - \frac{D_x \mathbf{x}'}{D} \right] [D_{xx} + D_x D'_x] &= D_{xx} + D_x D'_x - D_x D'_x = D_{xx} \\ &\Rightarrow D_{xx} [D_{xx} + D_x D'_x]^{-1} = \left[\mathbf{I} - \frac{D_x \mathbf{x}'}{D} \right] \end{aligned}$$

provides us with

$$\begin{bmatrix} 0 + CD'_x \frac{C^{-1}\mathbf{x}}{D} & CD'_x \left[C^{-1}[D_{xx} + D_x D'_x]^{-1} - C^{-1} \frac{\mathbf{xx}'}{D^2} \right] \\ CD_{xx} \frac{C^{-1}\mathbf{x}}{D} & D_x \frac{\mathbf{x}'}{D} + CD_{xx} \left[C^{-1}[D_{xx} + D_x D'_x]^{-1} - C^{-1} \frac{\mathbf{xx}'}{D^2} \right] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I} \end{bmatrix}.$$

Q.E.D.

Appendix 2: Derivation of economies of scope at the sample mean with mean scaled data

If all inputs and outputs are scaled so that their means are equal to one, then the log of these means are equal to zero. Hence the first derivatives simplify to

$$\frac{\partial D}{\partial x_i} = \alpha_i \quad \text{and} \quad \frac{\partial D}{\partial y_j} = \beta_j,$$

and the second derivatives become

$$\frac{\partial^2 D}{\partial x_i^2} = \alpha_i^2 - \alpha_i + \alpha_{ii}, \quad \frac{\partial^2 D}{\partial x_i \partial x_j} = \alpha_i \alpha_j + \alpha_{ij} \quad \text{and} \quad \frac{\partial^2 D}{\partial x_i \partial y_j} = \alpha_i \beta_j + \gamma_{ij}.$$

Thus we obtain

$$D_{xx} = \begin{bmatrix} \alpha_1^2 + \alpha_{11} - \alpha_1 & \cdots & \alpha_1 \alpha_n + \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_n \alpha_1 + \alpha_{n1} & \cdots & \alpha_n^2 + \alpha_{nn} - \alpha_n \end{bmatrix} \quad D_{yy} = \begin{bmatrix} \beta_1^2 - \beta_1 + \beta_{11} & \cdots & \beta_1 \beta_m + \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_m \beta_1 + \beta_{m1} & \cdots & \beta_m^2 - \beta_m + \beta_{mm} \end{bmatrix}$$

$$D_{xy} = \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_1 \beta_m + \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_n \beta_1 + \gamma_{n1} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \quad D_{yx} = \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_n \beta_1 + \gamma_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_1 \beta_m + \gamma_{1m} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix}.$$

Then using these expressions we have

$$\begin{aligned} & \left\{ D_y D_y' + D_{yx} \left[D_{xx} + D_x D_x' \right]^{-1} D_{xy} - D_{yy} \right\} = \\ & = \begin{bmatrix} \beta_1^2 & \cdots & \beta_1 \beta_m \\ \vdots & \ddots & \vdots \\ \beta_m \beta_1 & \cdots & \beta_m^2 \end{bmatrix} - \begin{bmatrix} \beta_1^2 - \beta_1 + \beta_{11} & \cdots & \beta_1 \beta_m + \beta_{1m} \\ \vdots & \ddots & \vdots \\ \beta_m \beta_1 + \beta_{m1} & \cdots & \beta_m^2 - \beta_m + \beta_{mm} \end{bmatrix} + \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_n \beta_1 + \gamma_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_1 \beta_m + \gamma_{1m} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \\ & \left\{ \begin{bmatrix} \alpha_1^2 + \alpha_{11} - \alpha_1 & \cdots & \alpha_1 \alpha_n + \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_n \alpha_1 + \alpha_{n1} & \cdots & \alpha_n^2 + \alpha_{nn} - \alpha_n \end{bmatrix} + \begin{bmatrix} \alpha_1^2 & \cdots & \alpha_1 \alpha_n \\ \vdots & \ddots & \vdots \\ \alpha_n \alpha_1 & \cdots & \alpha_n^2 \end{bmatrix} \right\}^{-1} \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_1 \beta_m + \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_n \beta_1 + \gamma_{n1} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \\ & = \begin{bmatrix} \beta_1 - \beta_{11} & \cdots & -\beta_m \\ \vdots & \ddots & \vdots \\ -\beta_{m1} & \cdots & \beta_m - \beta_{mm} \end{bmatrix} + \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_n \beta_1 + \gamma_{n1} \\ \vdots & \ddots & \vdots \\ \alpha_1 \beta_m + \gamma_{1m} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \left[\begin{bmatrix} 2\alpha_1^2 + \alpha_{11} - \alpha_1 & \cdots & 2\alpha_1 \alpha_n + \alpha_{1n} \\ \vdots & \ddots & \vdots \\ 2\alpha_n \alpha_1 + \alpha_{n1} & \cdots & 2\alpha_n^2 + \alpha_{nn} - \alpha_n \end{bmatrix} \right]^{-1} \begin{bmatrix} \alpha_1 \beta_1 + \gamma_{11} & \cdots & \alpha_1 \beta_m + \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_n \beta_1 + \gamma_{n1} & \cdots & \alpha_n \beta_m + \gamma_{nm} \end{bmatrix} \end{aligned}$$

Q.E.D.