Signalling quality with posted prices

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Abstract

We study a game in which the seller of an indivisible object wants to sell her object to a finite number of potential buyers with a posted price. The environment is such that the seller has some private information about the quality of the object that cannot be communicated with buyers at zero cost. We focus on the separating equilibrium of this game in which the seller signals her actual type via the posted price. The conditions of the existence and the uniqueness of this equilibrium are studied. In an example, we calculate the seller’s expected payoff at this equilibrium and further discuss some comparative statistics.

Keywords: Informed seller, posted price, signalling, asymmetric information.

JEL Classification: D40, D44, D81, D82.
1 Introduction

One of the most popular selling methods that has been practised in many real-world markets is the posted price, particularly the case in which the seller of an object posts a single price as a take-it-or-leave-it offer to a set of potential buyers. In the current paper, an environment has been described in which the seller of an object has private information about its quality. When the buyers care about this information and there is no costless method of communication between the seller and the buyers, we encounter the famous problem of adverse selection. In fact, we extend the classic model in Akerlof (1971), which first addressed this problem, to a more general case with continuous types of sellers to investigate how sellers behave in such markets.

There are numerous efforts to compare the posted price selling method with other selling mechanisms such as standard auctions to justify why sellers use posted prices while standard auctions are usually the revenue-maximising selling methods under the assumptions of the standard models. Indeed, Myerson (1981) shows with the independent private valuation (IPV) setting, a second-price auction with a suitable reserve price is the optimal selling mechanism for the seller of a single indivisible object. However, if we relax the assumptions of the IPV models, the optimal auction may not be the revenue-maximising method. Some papers in the literature discuss the advantages of posting a price for a monopolist. For example, Campbell and Levin (2006) show how a simple change in the classic IPV model could result in the advantage of a posted price mechanism compared to a standard auction. There could be several reasons for a seller to choose the posted price selling method, which are not the main concern of this study. Specifically, we are going to discuss the revenue-maximising price for an informed seller who chooses to sell her object with a posted price mechanism.

In game theory, the concept of Perfect Bayesian Equilibrium (PBE) has been extensively used for sequential move games with private information. This concept is also relevant to signalling games. In these games, the party that moves first usually conveys some private information about the true state of the world, while the other party only assigns some prior probability to the states. These signalling games usually have multiple PBE even with two types of the informed party. There have been many attempts to eliminate some of these PBEs to reduce the set of equilibria for these games. The most famous equilibrium refinement method, known as the Intuitive Criterion, is introduced by Cho and Kreps (1987). They restrict out-of-equilibrium messages to those that are reasonable and eliminate those equilibria with unreasonable out-of-equilibrium messages. Only one separating equilibrium in which the seller signals her true type can survive from their refinement method. Although the intuitive criterion is very important, but it
may not be helpful for more than two types of the informed party, which is the case in the current model. However, there is another criterion known as the Divinity Criterion introduced by Banks and Sobel (1987), which is more helpful when there are more than two types of informed players. This criterion restricts the attention to those types that have the highest likelihood of sending the out-of-equilibrium messages. These types are usually those in which the actions of the undertaker provide them with a payoff higher than the equilibrium. In an example with three types, Munoz-Garcia and Espinola-Arredondo (2011) show how the intuitive criterion fails to restrict the set of equilibria, but with the divinity criterion, only the most efficient separating equilibrium remains in the set of equilibria. In the current study, we focus on this unique separating equilibrium and will argue the incentive compatibility of it in a posted price mechanism.

Cai et al. (2007) study the reserve price signalling in a second-price auction with a setting closely related to ours. They argue that there exists a unique separating equilibrium in which the seller signals her actual type with a reserve price. This equilibrium goes through the lowest type of the seller, which sets the reserve price exactly like the full information case. The intuition for the stability of such equilibrium is that the seller with a low type does not choose a higher reserve price than the equilibrium reserve price because the probability of no sale is higher for higher reserve prices. Therefore, the marginal cost of increasing the reserve price is higher than the marginal expected gain for these types. Jullien and Mariotti (2006) is another study closely related to Cai et al. (2007), which studies the same auction game and its unique separating equilibrium with only two potential buyers. They argue that the probability of sale in the case of private information equilibrium is less than the symmetric information case.

In section 2, the model and the mechanism are presented. In section 3, the separating equilibrium analysis is studied. In section 4, an example investigates the implication of the equilibrium. Finally, section 5 concludes the remarks.

2 Model

There is a seller with an indivisible object for sale who faces a set of \( N = \{1, \ldots, n\} \) potential buyers. The setting is such that the seller chooses a price \( p \) as a take-it-or-leave-it offer to buyers. Potential buyers can either accept or reject this offer, and if multiple buyers accept the offer at \( p \), one of them wins the object randomly. The seller privately observes a signal \( s \) drawn from a known distribution \( F_0 \) with support \([0, s]\), which is twice-differentiable with a continuous density \( f_0 \). Each buyer \( i \) has a private signal \( x_i \) for the object, which is independently and identically distributed according to the distribution function \( F \) on \([0, \bar{x}]\), again
twice-differentiable and with continuous density $f$. The valuation of each buyer $i$ for the object is given by $V_i : [0, \bar{x}] \times [0, \bar{s}] \to \mathbb{R}_+$, a continuous function of her individual private signal $x_i$ as well as the seller’s signal $s$. The valuation function is increasing in both signals and symmetric for all buyers. Therefore, in this environment, a buyer cares not only about her own signal but also about the seller’s signal but not the other buyers’ signals. The seller’s own valuation of the object is given by $V_0 : [0, \bar{s}] \to \mathbb{R}_+$, a continuous and increasing function of her own signal. It is further assumed that the hazard rate function of $F$ is increasing.

The initial condition that needs to be satisfied for the optimal $p$ is that the posted price has to be greater than or equal to the seller’s valuation; otherwise, there is no rational explanation for the seller to sell the object. If $p \geq v_0(s)$, then $s \leq v_0^{-1}(p)$. The seller first announces the posted price, then buyers update their beliefs about the seller’s signal after observing the price. Suppose $\tilde{s}$ is what buyers believe the seller’s signal is, after they observe the posted price. Thus, each buyer’s expected value of the object with respect to the realisation of the seller’s signal is, $v(x_i, \tilde{s}) = E[V_i|s = \tilde{s}, s \leq v_0^{-1}(p)]$. According to the buyers’ expected values, only buyers with valuation $v(x_i, \tilde{s}) \geq p$ are willing to accept the price. Therefore, a buyer accepts $p$ if and only if her expected value is higher than $p$ given the belief about the seller’s signal. Considering the fact that buyers’ beliefs about the seller’s signal are in the same manner, define $x^*$ as the highest type of buyer that has an expected value such that she is indifferent to accepting or rejecting the posted price. A revenue-maximising price needs to assign zero expected payoff to a type of buyer with signal $x^*$. Thus, we have,

$$p = v(x^*, \tilde{s}),$$

which is the expected value of the type $x^*$.

The price that satisfies equation (1) is a function of $x^*$. Therefore, the seller’s expected revenue from posting this price becomes,

$$U(x^*, \tilde{s}) = \left(v(x^*, \tilde{s}) - v_0(s)\right)(1 - F_1(x^*)),$$

where $F_1(x^*)$ is the highest order statistic of $F$. We differentiate the argument in (2) with respect to $x^*$ to find the optimal posted price $^2$

$$D_1 U(x^*, \tilde{s}) = \frac{\partial v(x^*, \tilde{s})}{\partial x^*}(1 - F_1(x^*)) - f_1(x^*)(v(x^*, \tilde{s}) - v_0(s)),$$

$^1$ The hazard rate function of $F$ is defined by $\lambda(x) = \frac{f(x)}{1 - F(x)}$.

$^2$Given a function $H : \mathbb{R}^n \to \mathbb{R}$, $D_i H(x_1, \ldots, x_i, \ldots, x_n)$ represents the partial derivative of $H$ with respect to its $i$-th argument evaluated at the point $(x_1, \ldots, x_n)$. 

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= f_1(x^*)(v_0(s) - \Gamma(x^*, \tilde{s})), \tag{4}

where

\[ \Gamma(x^*, \tilde{s}) = v(x^*, \tilde{s}) - \frac{\partial v(x^*, \tilde{s})}{\partial x^*} \frac{1 - F_1(x^*)}{f_1(x^*)}. \]

Differentiating (\ref{eq:Gamma}) with respect to \( \tilde{s} \) gives us,

\[ D_2 U(x^*, \tilde{s}) = \frac{\partial v(x^*, \tilde{s})}{\tilde{s}} (1 - F_1(x^*)). \tag{5} \]

**Lemma 2.1.** As long as the hazard rate function of \( F \) is increasing, then \( \psi(x) = \frac{1 - F_1(x)}{f_1(x)} \) is decreasing in \( x \).

Proof. Appendix.

### 3 Separating Equilibrium

In this game, like most other signalling games, there are many equilibria. However, with a suitable standard equilibrium refinement method, we can eliminate most of these equilibria. Because the seller’s signal is continuous, the Divinity Criterion by Banks and Sobel (1987), can be applied here, for example, to eliminate all pooling and partial pooling equilibria. Our purpose is to determine the unique separating equilibrium in which the seller signals her true type via the posted price. The existence of such an equilibrium is justified by the results in Riley (1979) and Mailath (1987). In particular, the \( \Gamma \) function has to be strictly increasing in its first argument for the existence of the separating equilibrium. It is straightforward to show; as long as the hazard rate function of \( F \) is increasing and the \( v \) function is concave in \( x \), then the \( \Gamma \) is strictly increasing in \( x \). Then, the equilibrium goes through the seller with type zero who sets the posted price as in the full information case. In fact, this behaviour is similar to Cai et al. (2007)’s signalling game, where the seller signals her true type via the reserve price in a second-price auction. Before we continue with the seller’s optimal choice of the posted price in the separating equilibrium, let us discuss the situation as if it was the full information case. If it was a full information situation and the seller’s signal was publicly observable, then the optimal posted price would satisfy the following equation,

\[ p^*(s) = v_0(s) + \frac{1 - F_1(p^*)}{f_1(p^*)}. \tag{6} \]

Now, for the lowest type, \( s = 0 \), it must be the case that,
The full information analysis is mainly relevant for the behaviour of the lowest type. In fact, in the separating equilibrium, the lowest type sets the posted price, as it is the full information case.

If the seller signals her true type via the posted price in equilibrium, we have, $\bar{s} = s$. We are going to find the $x^*$ that maximises the seller’s payoff given the separating equilibrium behaviour. $x^*$ is increasing in $s$ because a higher seller’s signal would result to a lower probability of sale. Thus, to maximise the seller’s payoff, we have,

$$\frac{\partial U(x^*, s)}{\partial x^*} \frac{dx^*}{ds} + \frac{\partial U(x^*, s)}{\partial s} = 0$$

(8)

**Proposition 3.1.** The differential equation (8) characterises the unique separating equilibrium of the posted-price mechanism with a condition that the seller with the lowest type sets the price the same as she would in the full information case.

Proof. See Appendix.

As mentioned above, this signalling equilibrium goes through the type zero’s optimal posted price, which is the same as the full information case. Mailath (1987) argues the incentive compatibility of these types of equilibria in general, and as we showed in the appendix, his result is applicable to the current separating equilibrium. Thus, no type has the incentive to deviate from this equilibrium and sets a different posted price. The intuition suggests that the marginal cost of reporting a higher posted price is higher than the expected gain. Therefore, no seller would deviate from this equilibrium, and every type reports his or her true signal via the posted price.

Next we are going to analyse the effect of change in the number of buyers on the optimal posted price in the separating equilibrium. First, we should investigate how the $\Gamma$ function changes when $n$ changes.

**Lemma 3.1.** The following condition is sufficient for $\psi(x) = \frac{1 - F_1(x)}{f_1(x)}$ to be increasing in $n$.

$$n \ln F(x) \leq -1$$

Proof. Appendix.

It is important to consider that $\psi$ function is increasing in $n$ for some standard distributions such as uniform distribution, without any particular condition. However, lemma 3.1 gives us a sufficient condition for any given distribution to have that property.
Proposition 3.2. The optimal posted price is higher for a higher number of potential buyers as long as $\psi$ function is increasing in $n$.

Proof. See Appendix.

To prove this proposition, it is enough to show that the $x^*$ increases by the number of buyers. Then, with higher $x^*$, the optimal posted price, which is the expected value of the type $x^*$ would also become higher. In the next section, we calculate the optimal posted price and the seller’s net expected payoff, in an example with linear valuations and uniform signals.

4 Example

As an example, suppose the valuations of the seller and the prospective buyers are linear functions of the signals. The seller’s valuation is a linear function of her signal $v_0(s) = \gamma s$ for $\gamma > 0$. Each buyer’s valuation is also a linear function of her own private signal plus the seller’s signal: $v(s, x_i) = s + x_i$.

According to the buyers’ expected valuations, only buyers with an expected value $v(\hat{s}, x_i) \geq p$ are willing to buy given the perception that the seller’s signal is $\hat{s}$. Because we focus on the separating equilibrium as discussed in the previous section, in equilibrium, the seller signals her true type with a posted price equal to the expected value of the buyer with type $x^*$. The buyer with this type must have zero expected payoff from buying the object at price $p$. To calculate $x^*$, it is first necessary to calculate the differential equation in (8). With our assumption for the valuations, we have,

$$f_1(x^*)(\gamma s - s - \Gamma(x^*))\frac{dx^*}{ds} + (1 - F_1(x^*)) = 0, \quad (9)$$

where

$$\Gamma(x^*) = x^* - \frac{1 - F_1(x^*)}{f_1(x^*)}.$$ 

We can rewrite this differential equation as follows,

$$\frac{dx^*}{ds} = \frac{1 - F_1(x^*)}{f_1(x^*)\Gamma(x^*) + (1 - \gamma)s}. \quad (10)$$

Let’s further suppose $n = 2$ and all signals are independent and distributed uniformly on $[0, 1]$. First of all, it is easy to check that $\Gamma(.)$ is a strictly increasing function. Thus, to find $x^*$ that characterises the separating equilibrium for every $s$, we need to solve the following differential equation.
\[ dx^* = \frac{1 - (x^*)^2}{2x^*(\frac{3x^2 - 1}{2x^2} + (1 - \gamma)s)}. \]  

To find the seller’s expected payoff at the separating equilibrium, we need to substitute \( x^* \) from (11) into the seller’s expected payoff function. Figure 1 shows the change in the seller’s net expected payoffs when \( \gamma \) changes. In fact, higher gammas show the higher value that the seller gives to her own signal. Thus, she values the object more, and therefore, her net expected payoff is lower for higher gammas. It is easy to show that the equilibrium expected payoff for the seller is decreasing in her own signal.

Figure 1: Net expected payoffs to the seller

Figure 2 shows how the minimum buyer type changes when the seller’s signal changes. By definition, we know the minimum type is increasing in the seller’s signal, but the figure shows that it is also concave for the current example.

Figure 3 shows the optimal posted price for every type. For the current example, because the valuation function of the buyer is linear, it is the sum of the minimum type plus the seller’s signal. Therefore, the posted price is also increasing in the seller’s signal.

The next step is to check how the seller’s expected payoff and the optimal posted price change when the number of potential buyers increases. From proposition 3.2 we know the minimum buyer type is increasing in the number of buyers, and therefore, the price is also increasing. Figure 4 shows the change in the seller’s expected payoffs and the posted price when the number of buyers increases to five and ten. When the seller’s signal becomes higher, because the differences between
the minimum type buyers become lower, the differences between posted prices also become very small. Thus, the optimal posted price of a seller with a very high type react very little to the change in the number of buyers.

5 Conclusion

We study the behaviour of a seller who has a unique object and private information about its quality. In this case, when there is no costless way to access the seller’s information, signalling is one credible method to reveal the seller’s true type. We show how a seller who chooses a posted price to sell her object signals her true type via the price. We argue that this separating equilibrium is the most stable one within the set of equilibria according to the divinity criterion. In an
example with linear valuations, we show that the seller’s net expected payoff is decreasing in her type, and the more she cares about her own signal, the less her expected payoffs would be. We further increase the number of potential buyers and show how it effects the optimal posted price and the seller’s expected payoffs.
Proof of Lemma 2.1
When the hazard rate function is increasing then,

\[ \frac{1 - F(x)}{f(x)}, \]  

is decreasing. We have,

\[ \frac{1 - F_1(x)}{f_1(x)} = \frac{1 - F^n(x)}{nF^{n-1}(x)f(x)} = \frac{F^{n-1}(x) + \ldots + F(x) + 1}{nf^{n-1}(x)} \left[ 1 - \frac{F(x)}{f(x)} \right] \]  

\[ = \left[ \frac{1}{n} + \frac{1}{nF(x)} + \ldots + \frac{1}{nF^{n-1}(x)} \right] \left[ 1 - \frac{F(x)}{f(x)} \right]. \]

The left hand bracket is clearly decreasing in \( x \). The right hand is also decreasing because of the increasing hazard rate. \( \square \)

Proof of Lemma 3.1

\[ \psi(x) = \frac{1 - F_1(x)}{f_1(x)} = \frac{1 - F^n(x)}{nF^{n-1}(x)f(x)}, \]

\[ \psi(x) = \frac{1}{f(x)} \left[ \frac{1}{nF^{n-1}(x)} - \frac{F(x)}{n} \right], \]

\[ \frac{\partial \psi(x)}{\partial n} = \frac{1}{f(x)} \frac{\partial}{\partial n} \left[ \frac{1}{nF^{n-1}(x)} - \frac{F(x)}{n} \right]. \]

As long as the first term in bracket is increasing in \( n \) the whole term is increasing in \( n \). Therefore, we have,

\[ \frac{\partial}{\partial n} \left( \frac{1}{nF^{n-1}(x)} \right) = \frac{-F^{n-1}(x) - n \ln F(x)F^{n-1}(x)}{(nF^{n-1}(x))^2}. \]

For this to be positive we must have,

\[ n \ln F(x) \leq -1. \]

\( \square \)

Proof of Proposition 3.1
First, rewrite the differential equation as follows

\[ \frac{dx^*}{ds} = \frac{D_2 U(x^*, s)}{D_1 U(x^*, s)}, \]  

(14)
substituting $D_1$ and $D_2$ from the differentiation of the expected payoffs, we have

$$\frac{dx^*}{ds} = -\frac{\partial v}{\partial x}(1 - F_1(x^*)) \frac{dF_1(x^*)}{ds} (v_0(s) - \Gamma(x^*, s))$$

(15)

With the assumption for the $\Gamma$ function, the right-hand side in (15) is always negative. Therefore, according to [Riley, 1979], there exists a unique solution going through the optimum of the seller with type zero in the full information situation that is strictly increasing.

We need to further show the incentive compatibility condition. The buyers’ beliefs about the seller’s signal can be written as $\tilde{s} = x^*(s)$. Therefore, the incentive compatibility condition is,

$$s = \arg\max \tilde{s} U(x^*(\tilde{s}), \tilde{s}),$$

(16)

and therefore, the equilibrium payoff is,

$$U(x^*(s), s) = \max \tilde{s} U(x^*(\tilde{s}), \tilde{s}).$$

(17)

If we rewrite the expected payoff function with respect to $x^*$ and differentiate it, then we have,

$$\frac{dU(x^*, x^*(s))}{dx^*} = D_1 U(x^*, x^*(s)) + D_2 U(x^*, x^*(s)) \frac{ds}{dx^*}. 

(18)

Thus, there is no incentive for a seller of a given type to deviate from this equilibrium.

□

Proof of Proposition 3.2

First let’s check how the lowest seller type reacts to an increase in the number of buyers. For the lowest type the equilibrium condition suggests,

$$v_0(0) = p^* - \frac{1 - F_1(p^*)}{f_1(p^*)}. 

(19)

Suppose $n$ increases, then since $\psi$ function is increasing in $n$ and $v_0(0)$ is independent of $n$. Then $p^*$ must increase to satisfy the above condition. Therefore, the lowest type would definitely increase the posted price when $n$ increases.

Rewrite the equilibrium differential equation in (15) as follows,

$$\frac{\partial x^*}{\partial s} = \frac{\frac{\partial v}{\partial x}(1 - F_1(x^*))}{\Gamma(x^*, s) + v_0(s) \frac{1 - F_1(x^*)}{f_1(x^*)}}$$

(20)
It is easy to check that $\Gamma$ is decreasing in $n$. The second term in the right hand side is the $\psi$ function and increasing in $n$. Now for a given $s$ if $n$ increases and $x^*$ doesn’t change the right hand side of the equality will become larger. Since $\Gamma$ is increasing in $x^*$ and $\psi$ function is decreasing in $x^*$, therefore, $x^*$ must increase to satisfy the equilibrium differential equation.

□
References


