Equilibrium Indeterminacy in a Model of Constrained Financial Markets

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Abstract

We present a General Equilibrium model with incomplete markets in which assets pay in units of a single good. In our model, agents are constrained to negotiate the same amount of assets at different states of the world. Differently from the standard result of economies with real assets, our model shows the existence of real indeterminacy. The presence of multiple equilibria raises issues related to choices of a desired equilibrium outcome.

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Keywords: Incomplete markets, equilibrium indeterminacy, real assets, trade restrictions.

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1 Introduction

Market incompleteness represents limitations on the trade of goods and assets for all possible future contingencies. Examples of these restrictions are transaction costs, asymmetry of information and coordination problems, among others. Because these situations are very frequent, the theory of general equilibrium with incomplete markets (GEI) has been extensively used in economic models.

In GEI models the number of available contracts is not sufficient to provide transfer of wealth from one to any other possible state of the world. However, this standard limitation on wealth transfer does not fully represent some types of contracts, such as retirement and college tuition plans. Besides the traditional GEI constraints on the structure of the matrix of assets’ returns, these contracts require restrictions on the amount of assets purchased. In many 401K retirement contracts, for example, the future retiree agrees to have the same share of his wage deducted each month. Usually the plans are designed by agreeing not only on the investor’s future contributions to the fund but on the composition of its portfolio. The 529 pre-paid tuition plans have a similar form. Often sponsored by state governments, these plans allow investors to purchase tuition credits for later use at a listed college or university. The amount of tuition credits is defined at the moment of enrollment in the plan, which means that investors lock in their choice of credits to receive in the future. A clear advantage is that these funds’ earnings are subject neither to federal taxes nor, in some cases, to state taxes. Besides tax incentives, some consumers perceive these types of investment to be good mechanisms for committing themselves to a systematic saving scheme.

We model exactly this type of situations, in which agents agree on the quantity of traded assets before they learn their real returns. Formally, the amount of assets to be traded has to be the same at different states of the world. We show that economies with these additional constraints exhibit a higher degree of price indeterminacy. This indeterminacy is real in the sense that for each price normalization there is a different equilibrium allocation. The importance of this result is that policies affecting prices of constrained assets have distributional effects. These policies represent an alternative to the traditional realization of transfers.

Real indeterminacy is not a concern in economies with numéraire assets, as shown by Geanakoplos and Polemarchakis (1986). However, when assets are nominal, i.e., delivering units of account, there are multiple equilibria, as illustrated by Cass (1985). The result of Cass (1985) inspired further investigation into nominal assets. Geanakoplos and Mas-Colell (1989) show the existence of real indeterminacy in a two-period economy.
when the number of consumers is sufficiently large. In independent work, Balasko and Cass (1989) analyze indeterminacy by allowing the economy to have more than two periods. Multiple equilibria occur even if only one group of agents does not have access to complete markets, as in Balasko et al. (1990). Werner (1990) proved that indeterminacy is preserved when there are more than two periods and assets can be retraded. Pietra (1992) relaxes the assumption of nominal assets by allowing the trade of assets with mixed returns. In a two-period model, economies with mixed assets have one more indeterminate price than economies with exclusively nominal assets. In our model, assets deliver in units of the single good, which means that our result of indeterminacy does not rely on the presence of nominal assets but on the combination of incomplete markets and additional restrictions.

Our restrictions are similar to those proposed by Polemarchakis and Siconolfi (1998). They show indeterminacy in economies where payments of goods are deferred. Here, we are concerned about restrictions on the trade of assets, not commodities. Malaith et al. (2004) consider constraints on the timing of decisions. In their model, the amount of labor is chosen before the realization of housing prices. As a consequence, investments in labor are inefficient. In contrast, we use a pure exchange economy model, where efficiency is not an issue.

We consider a finite horizon exchange economy with at least three periods of time. Besides additional restrictions on the trade of assets, other assumptions are standard in GEI models. Section 2 presents our model and the main result. Two examples are discussed in Section 3. Proofs are left to the Appendix.

2 The Model

There is only one good in a finite-horizon economy with at least three periods of time \((t = 1, ..., T, T \geq 3)\). States of the world are \(s = 1, ..., S\). There is a finite number of agents, indexed by \((i = 1, ..., I)\) and endowed with \(\omega^i \in \Omega^i = \mathbb{R}_{++}^S\). Define the set of endowments as \(\Omega = \Omega^1 \times ... \times \Omega^I\). Individuals have contingent preferences represented by utility functions \(u^i_s\), mapping \(\mathbb{R}_{++}\) onto \(\mathbb{R}_+\). We define the expected utility function \(U^i\) as:

\[
U^i = \sum_{s \in S} \pi^i_s u^i_s, \text{ where:}
\]

\(\pi^i_s\) is the subjective probability of occurrence of \(s\).

**Assumption 1.** The function \(u^i\) satisfies

i. continuity and smoothness: \((u^i \text{ is } C^\infty \text{ on } \mathbb{R}_+)\).
ii. boundary condition: \( \text{cl}\{x \in X^i : u(x) \geq u(y)\} \subset X^i_{++} \), \( \forall y \in X^i_{++} \), where \( X^i \) is the commodity space of agent \( i \) and \( \text{cl} \) denotes the closure of a set.

iii. monotonicity: for two commodity bundles \( x \) and \( y \), \( x > y \Rightarrow u^i(x) > u^i(y) \).

iv. strict concavity: for any \( x, y \in \mathbb{R}_+ \), \( x \neq y \), and \( \alpha \in [0,1] \), \( u(\alpha x + (1 - \alpha)y) > \alpha u(x) + (1 - \alpha)u(y) \).

Agents have access to short-lived assets: payments of obligations occur one period after assets are traded. The structure of the financial markets is represented by the matrix \( W \) of prices \(-q \in \mathbb{R}^J\) and payoffs. The dimension of \( W \) is \( S \times J \).

**Assumption 2.**

i. numéraire assets: payoffs are defined in amounts of the single commodity.

ii. no-redundancy: \( \text{rank } W = J \).

iii. no-arbitrage: there is no vector \( \alpha \in \mathbb{R}^J \) satisfying \( W\alpha > 0 \).

Individuals decide about their demands for goods, \( x^i \in X^i \), and for assets, \( z^i \in \mathbb{R}^J \).

At every state of the world, we normalize the commodity price as one. The set of agent \( i \)'s budget constraints is:

\[
\omega^i - x^i + Wz^i \geq 0 \tag{2.1}
\]

Besides the budget constraints, all agents face the same additional restrictions on the trade of assets. When deciding to trade \( z^i_s \) units of asset \( s \), individual \( i \) must also trade \( \alpha z^i_{s'} \) units of asset \( s' \). We represent these restrictions as:

\[
Rz^i = 0, \tag{2.2}
\]

where \( R \) is a \( \tau \times J \) matrix with entries equal to 1, \(-\alpha\), or 0. Then, (2.2) corresponds to \( \tau \) additional restrictions like \( z^i_s = \alpha z^i_{s'} \), where \( s \) and \( s' \) are two arbitrary different states of the world, and \( \alpha \in \mathbb{R} \). Without loss of generality, we fix \( \alpha = 1 \). Each column of \( R \) defines one restriction.

**Definition 1. Past History**

The past history of state \( s \) is the set \( S_{-s} \) of all states of the world that occur before the realization of \( s \).

Our result depends on some assumptions on the restrictions defined by \( R \):

**Assumption 3.**

i. diversity: The number of agents is greater than the number of restrictions \( \tau \).
ii. matrix $R$: restrictions (2.2) affect all assets sold at period $T - 1$. Assets sold at the same state of the world are subject to different restrictions. If two assets, $j$ and $k$, face the same restriction $z_j = z_k$, they are negotiated in states of the world that have the same past history.

Assumption 3.i guarantees enough diversity among agents. Assumption 3.ii eliminates cases in which restrictions affect assets bought at the same state of the world. If that was the case, they could be merged into only one asset.

Let $B^i$ be the set of $S$ allocations $(x^i, z^i)$ satisfying their respective budget constraints and $C$ be the set of $J$ assets satisfying the $\tau$ additional restrictions:

$$
B^i = \{(x^i, z^i) \mid \omega^i - x^i + Wz^i \geq 0\}
$$

$$
C = \{z^i \mid Rz^i = 0\}
$$

Agents choose a pair of goods and assets $(x^i, z^i)$ that lies in $B^i \cap C$.

**Definition 2. Constrained Financial Equilibrium (CFE)**

A constrained financial equilibrium is a collection of allocations $(\bar{x}, \bar{z}) \in \mathbb{R}^{SI} \times \mathbb{R}^{JI}$ and prices $\bar{q} \in \mathbb{R}^J$ that satisfies:

i. $(\bar{x}^i, \bar{z}^i) \in \arg \max \{U^i(x^i) \mid (x^i, z^i) \in B^i \cap C\} \forall i = 1, ..., I$.

ii. $\sum_{i=1}^I \bar{x}^i_j = 0 \forall j = 1, ..., J$.

**Proposition 1. Existence of the CFE.**

There exists a constrained financial equilibrium.

The next definition formalizes the concept of real indeterminacy, which occurs when the equilibrium price vectors are not locally isolated.

**Definition 3. Real Indeterminacy**

Consider an economy with initial allocation $\omega = (\omega^1, ..., \omega^i, ..., \omega^I)$. There is real indeterminacy if and only if we can find a one to one $C^1$ function that maps an open subset of $\mathbb{R}^d$ to the set of CFE allocations, where $d$ is a positive integer. The value of $d$ determines the degree of real indeterminacy for this economy.

Intuitively, the degree of real indeterminacy corresponds to the maximum number of prices whose change in direction affects the equilibrium allocation.

**Proposition 2. Existence of Real Indeterminacy**

Consider an economy with at least two assets $j$ and $k$ subject to restriction (2.2). Let the number of assets paying at $T - 1$ be strictly lower than the number of states of the world at $T - 1$. Then, there exists a set of full measure in $\bar{\Omega} \in \Omega$ such that, if $\omega \in \bar{\Omega}$, this economy presents real indeterminacy.
Proposition 2 shows that additional constraints like (2.2) are not enough for the occurrence of real indeterminacy. Indeterminacy prevails only in the cases where agents cannot insure for all states that precede the state in which they exchange the restricted asset. Incomplete markets lack the instruments of price adjustment because the number of assets available does not allow for transfer of wealth in all states of the world. The next Section illustrates exactly this point, where we contrast complete to incomplete markets economies.

3 Examples

In this section, two examples show how the same kind of restriction can have different effects, depending on the financial structure of the markets. In both examples spot prices of the single commodity are normalized to 1. They are illustrated by the following event-tree:

The above event-tree can represent, for example, 529 pre-paid tuition plans. At period 1 agents enroll in the plan, agreeing on the amount of tuition purchased. Tuition credits are received at period 2. At this period, the return of attending to college is subject to uncertainty. If states invested in universities\(^1\), attending to college pays a higher return than otherwise. Because the amount of tuition credit is agreed to be the same, independently on the investments directed to state universities, we need to impose an additional restriction, that is, the amount of assets bought at states 2 and 3 is the same.

The two examples that follow show the role of market incompleteness on the existence of multiplicity of equilibrium. We start with the case of a complete markets economy.

\(^1\)Most of the institutions listed in 529 pre-paid tuition plans are state universities.
3.1 Incomplete Markets

Assume now that individuals cannot insure against all possible states of the world. As an example, consider the case in which there is no transfer of wealth from period 1 to 2 and, from 2 to 3, there is one asset paying one unit at state 4 and another paying one unit at state 5. The event tree is as represented above. Individuals are supposed to trade the same amount of assets at both states of period one. Assets are indexed according to the state of the world they pay. The set of assets is \{4, 5\}.

In this economy additional restrictions have consequences on allocation of the consumption good. Since some prices are allowed to vary freely, changes in those prices modify budget constraints in such a way that it is not possible to satisfy all restrictions by having the same distribution of excess demand among states of the world.

The set $B^i \cap C$ determines that, in equilibrium the following must hold:

\[
\begin{align*}
    w^i_1 - x^i_1 &= 0 \\
    w^i_2 - x^i_2 - q_4 z^i_4 &= 0 \\
    w^i_3 - x^i_3 - q_5 z^i_4 &= 0 \\
    w^i_4 - x^i_4 + z^i_4 &= 0 \\
    w^i_5 - x^i_5 + z^i_4 &= 0
\end{align*}
\]

(3.1)

As argued previously, additional restrictions create linear dependence between market clearing conditions of assets sold at date-events 2 and 3. Either $q_4$ or $q_5$ are exogenously determined. Let $\mathbf{q}$ be the equilibrium price vector associated with the equilibrium allocation $(\mathbf{x}, \mathbf{z})$.

Consider a change in the choice of $q_5$ satisfying $q_5 = \theta q_5$. The first three equations of 3.1 become:

\[
\begin{align*}
    w^i_1 - x^i_1 &= 0 \\
    w^i_2 - x^i_2 - q_4 z^i_4 &= 0 \\
    w^i_3 - x^i_3 - \theta q_5 z^i_4 &= 0
\end{align*}
\]

(3.2) (3.3) (3.4)

From 3.4 it must be that, for $z^i_4 \neq 0$, the only value of $\theta$ that keeps allocation as before is one.
3.2 Complete Markets

Consider an economy that provides full insurance. For each state of the world there is one asset that pays one unit of the consumption good at that date-event and nothing where else. In this example $S - 1 = 4$ and the matrix of returns is a row of zeros followed by a $4 \times 4$ identity matrix. Assets are indexed according to the state of the world they pay. So, we define the demand of agent $i$ for the asset that pays in state 2 as $z^i_2$. The set of assets is $\{2, 3, 4, 5\}$.

Agents face the following additional restrictions:

$$z^i_4 = z^i_5 \text{ for all agents } i. \quad (3.5)$$

When spot prices are normalized to be 1 at every state of the world, the set $B^i \cap C$ can be represented by the following system of equations:

$$B^i \cap C = \left\{ (x^i, z^i) : \begin{array}{l}
w^i_1 - x^i_1 - q_2 z^i_2 - q_4 z^i_4 = 0 \\
w^i_2 - x^i_2 + z^i_2 - q_4 z^i_4 = 0 \\
w^i_3 - x^i_3 + z^i_3 - q_5 z^i_5 = 0 \\
w^i_4 - x^i_4 + z^i_4 = 0 \\
w^i_5 - x^i_5 + z^i_5 = 0 
\end{array} \right\}$$

As stated in Definition 2.ii, market clearing conditions are:

$$\sum_{i=1}^{I} z^i_2 = 0, \sum_{i=1}^{I} z^i_3 = 0, \sum_{i=1}^{I} z^i_4 = 0, \sum_{i=1}^{I} z^i_5 = 0. \quad (3.6)$$

Since agents are solving their maximization problems constrained to the set $B^i \cap C$, there is linear dependence between some market conditions. Note from (3.6) that, once $\sum_{i=1}^{I} z^i_4 = 0$, equation (3.5) implies $\sum_{i=1}^{I} z^i_5 = 0$. Then, one price is sufficient to clear these two markets; the other is free to vary. However, the free choice of exogenous prices has no effect on allocation. In other words, agents’ choices do not depend on this price.

Let $\pi^i_s$ be the subjective probability of state $s$ and consider the Lagrangean function derived from agents’ maximization problem when they have a utility function that is

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2From monotonicity, budget constraints are considered with equality.
separable among states of the world:

\[
\mathcal{L} = u_i^1(x_i^1) + \sum_{s=2}^{s=5} \pi_s^i u_s^i(x_s^i)
+ \mu_1^i(w_1^i - x_1^i - q_2^i z_2^i - q_3^i z_3^i)
+ \mu_2^i(w_2^i - x_2^i + z_2^i - q_4^i z_4^i)
+ \mu_3^i(w_3^i - x_3^i + z_3^i - q_5^i z_4^i)
+ \mu_4^i(w_4^i - x_4^i + z_4^i)
+ \mu_5^i(w_5^i - x_5^i + z_5^i).
\]

The correspondent first order conditions are listed below:

\[
\begin{align*}
u_1'(x_1) &= \mu_1^i \\
\pi_2^i u'(x_2) &= \mu_2^i \quad \pi_3^i u'(x_3) = \mu_3^i \\
\pi_4^i u'(x_4) &= \mu_4^i \quad \pi_5^i u'(x_5) = \mu_5^i \\
\mu_1^i q_2 &= \mu_2^i \\
\mu_3^i q_3 &= \mu_5^i
\end{align*}
\] (3.7)

\[\mu_2^i q_4 + \mu_3^i q_5 = \mu_4^i + \mu_5^i \] (3.8)

Substitution of 3.7 into 3.8 yields:

\[\mu_1^i q_2 q_4 + \mu_1^i q_3 q_5 = \mu_4^i + \mu_5^i \] (3.9)

Since linear dependence of market clearing conditions allows for free choice of one out of prices \(q_4\) and \(q_5\), let \(q_5\) be this price and leave \(q_4\) to attain market clearing. Different choices of \(q_4\) do not change the equilibrium allocation. Equation 3.9 is an equilibrium condition for every individual \(i\). Using an upper bar to indicate equilibrium variables, it can be rewritten as:

\[
\bar{q}_4 = a^i - b^i \bar{q}_5, \text{ where:}
\]

\[
a^i = \frac{\pi_1 + \pi_5}{\pi_1 \bar{q}_2}, \quad b^i = \frac{\bar{q}_3}{\bar{q}_2}.
\] (3.10)

Let \(q_5\) vary such that it moves from \(\bar{q}_5\) to \(\gamma \bar{q}_5\). Considering this variation, \(q_4\) must satisfy:

\[q_4 = a^i - \gamma b^i \bar{q}_5 \quad \forall \ i. \] (3.11)

Note that 3.11 must hold for every individual \(i\). If allocations are to remain the same as in the previous equilibrium, the first order conditions determine that neither \(a^i\) nor \(b^i\) can vary. Movements in \(b^i\) are equivalent to changes in \(\bar{q}_2\), \(\bar{q}_3\), or both. From (3.7), changes in \(q_2\) imply a different marginal rate of substitution from state 2 to 1. Similarly, a different value of \(q_3\) determines different allocation in states 1 and 3. Similarly, if \(a^i\)
varies, the set of first order conditions determines different levels of consumption.

There is a price $q_4$ for which individuals will continue to maximize their consumption if and only if 3.11 is equal for every individual $i$:

$$q_4 = a^i - \gamma b^i \bar{q}_5 = a^k - \gamma b^k \bar{q}_5$$  \hspace{1cm} (3.12)

$$= \bar{q}_4 + (1 - \gamma) b^i \bar{q}_5 = \bar{q}_4 + (1 - \gamma) b^k \bar{q}_5, \forall i, k.$$ \hspace{1cm} (3.13)

The previous equality holds if and only if $b^i = b^k$. But $b^i = \frac{\bar{q}_3}{\bar{q}_2} = b^k$ is the same for every individual $i$. Changes in the fixed price $q_5$ are followed by movements in $q_4$ that keep the same equilibrium allocation. In this example, $q_4 = \bar{q}_4 + (1 - \gamma) \frac{\bar{q}_3 \bar{q}_5}{\bar{q}_2}$ when $q_5$ moves from $\bar{q}_5$ to $\gamma \bar{q}_5$.

The key point is that individual’s first order conditions determine a linear relation among prices of assets with restricted trade. Since markets are complete, this relation is the same for all individuals. There are no real effects.

4 Appendix

Proof. Proof of Proposition 1

Consider the set $\mathcal{B}^i \cap \mathcal{C}$ of restrictions faced by individual $i$. Because $\mathcal{C}$ is the same for everyone, the matrix $W$ can incorporate the additional restrictions in (2.2).

Construct a $J \times (J - \tau)$ matrix, called $M$, as follows. If asset 1 is non-restricted, the first column of $M$ has a unit entry in the first row and 0 everywhere else. If asset 1 is restricted, then $M$ has a unit entry at the first row and at every $j^{th}$ row corresponding to assets $j$ with $z_1 = z_j$. The other entries in this column are equal to 0. A similar rule applies to the second column: if asset 2 is unrestricted, there is a unitary entry at $m_{2,2}$ and 0 everywhere else. If the restriction $z_2 = z_j$ holds, no column is added to matrix $M$. If restrictions $z_2 = z_j$ with $j > 2$ hold, then the second and $j^{th}$ rows have a unitary entry, while the remaining rows have entries equal to 0. Proceed in this fashion to check for the remaining assets. One more column is added only if there is no previous unitary entry at the $j^{th}$ row.

Let $\tilde{W}$ be the transformed matrix, defined as $\tilde{W} = WM$. $M$ is a matrix that merges all assets sharing a restriction in $R$. More specifically, while every column in $W$ corresponds to one single asset, assets that face a common restriction enter at the same
column in matrix $\tilde{W}$. For example, with the following financial structure:

$$W = \begin{pmatrix}
-q_1 & 0 & 0 \\
1 & -q_2 & 0 \\
1 & 0 & -q_3 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

and the restriction $z_2 = z_3$, we have:

$$M = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad \tilde{W} = WM = \begin{pmatrix}
-q_1 & 0 \\
1 & -q_2 \\
1 & -q_3 \\
0 & 1 \\
0 & 1
\end{pmatrix}$$

The remaining of this proof follows the steps proposed by Villanacci et al. (2002). First, define individual $i$’s demand as a function that maps the set of prices $q$ and endowments $\omega$ to the set of commodities after solving:

$$(x^i, z^i) \in \arg \max \{U^i(x^i)|(x^i, z^i) \in B^i \cap C}\forall i = 1, ..., I. \quad (4.1)$$

To make notation simpler, we suppress superscripts $i$. Let $DU$ denote the Jacobian of $i$’s utility function and $\mu$ the vector of Lagrange multipliers. The first order conditions are:

$$DU(x) - \mu = 0$$
$$\omega - x + \tilde{W}z = 0$$
$$\mu \tilde{W} = 0 \quad (4.2)$$

Because both $B$ and $C$ are compact sets, demand functions exist. Assumption 1.iv of strict concavity guarantees their uniqueness.

The definition of the CFE includes $J$ market clearing conditions. However, some of them are redundant, because all consumers face the same restrictions, determined by the matrix $R$. Given the restriction $z_j = z_k$, for example, once $\sum_{i=1}^I z_j^i = 0$ is satisfied, we also have $\sum_{i=1}^I z_k^i = 0$.

Since there are $\tau$ redundant equations, whenever agents face the restriction $z_j = z_k$, we fix $\tau$ prices by setting $q_j = q_k$. Let $\tilde{J}$ be the set of all unrestricted assets and all assets $j$ for which there is no restriction $z_j = z_k$, $k < j$. For instance, in the above example, $\tilde{J} = \{1, 2\}$. The solution to the CFE is represented by the following system
of equations, where the first three sets of equations must be satisfied by all consumers \( i \in I \):

\[
\begin{align*}
DU(x) - \mu &= 0 \\
\omega - x + \tilde{W}z &= 0 \\
\mu'\tilde{W} &= 0 \\
Rq &= 0 \\
\sum_{i=1}^{I} z_{j}^i &= 0 \quad \forall j \in \tilde{J}
\end{align*}
\] (4.3)

Define a function that maps the sets of commodities allocations \( x \), Lagrange multipliers \( \mu \), assets allocations \( z \), assets prices \( q \) and endowments \( \omega \) to the left hand side of the system (4.3). The domain of \( F \) is \( \Xi = X^1 \times \ldots X^{i} \times \ldots X^I \times \mathbb{R}_{++}^{S_I} \times \mathbb{R}^{J_I} \times \Omega^1 \times \ldots \Omega^i \times \ldots \Omega^I \) and its codomain is a set of real numbers with the same dimension as \( \Xi \). In summary, \( F : \Xi \rightarrow \mathbb{R}^{\dim \Xi} \).

Let \( x_o \) be a Pareto optimal allocation and \( \rho \) be an arbitrary number on the interval \([0, 1]\). The technique proposed by Villanacci et al. (2002) is to choose a function \( G \) and construct an homotopy \( H \) from \( F \) to \( G \). Then, it suffices to show that: (i) there is a unique solution to \( G=0 \); (ii) the Jacobian of \( G \), evaluated at the point that solves \( G=0 \), has full rank; and (iii) the set of solutions to \( H \) is compact.

The suggested homotopy substitutes the original endowments in \( F \) by a combination between initial endowments and the Pareto optimal allocation \( x_o \). For \( \rho \in [0, 1] \), the homotopy \( H : \Xi \times [0, 1] \rightarrow \mathbb{R}^{\dim \Xi} \) is:

\[
\begin{align*}
DU(x) - \mu &= 0 \\
(1 - \rho)\omega + \rho x_o - x + \tilde{W}z &= 0 \\
\mu'\tilde{W} &= 0 \\
Rq &= 0 \\
\sum_{i=1}^{I} z_{j}^i &= 0 \quad \forall j \in \tilde{J}
\end{align*}
\] (4.4)

The above system of equations is equal to \( F \) when \( \rho = 0 \) and to \( G \) when \( \rho = 1 \).

From assumption 1.iv of strict concavity, there is a unique solution to \( G \), denoted as \( \xi_o \).

Moving to the second step of the proof, we need to show that \( DG(\xi_o) \) is a full rank matrix, where \( DG(\xi_o) \) is the derivative of \( G \) evaluated at \( \xi_o \). If \( DG(\xi_o) \) is full rank, its dimension is equal to \((2S + J)I + J\). From (4.4), we can calculate \( DG(\xi_o)\Delta \), where \( \Delta = (dx \ d\mu \ dz \ dq_R \ dq_U)' \), \( W_q \) is a matrix that replaces all price entries of \( W \) by one, and has zero entries otherwise. \( q_R \) is the price vector of restricted assets and \( q_U \) is the

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\(^3\text{For a definition of homotopy, see Villanacci et al. (2002), p. 160.}\)
price vector of unrestricted assets. Using the fact that \( z = 0 \) at \( \xi_o \), we have:

\[
DG(\xi_o) \Delta = \begin{bmatrix}
D^2 U(x) & -I & 0 & 0 & 0 \\
-I & 0 & \tilde{W} & 0 & 0 \\
0 & \tilde{W} & 0 & -\mu W_q & -\mu W_q \\
0 & 0 & 0 & R & 0 \\
0 & 0 & I & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
dx \\
d\mu \\
dz \\
dq_R \\
dq_U \\
\end{bmatrix}
\]  (4.5)

We want to show that \( DG(\xi_o) \Delta = 0 \) implies \( \Delta = 0 \).

Suppose \( DG(\xi_o) \Delta = 0 \) but \( \Delta \neq 0 \).

If all \( dx^i = 0 \), (4.5) implies \( \Delta = 0 \). Then, let \( dx^i \neq 0 \) for at least one agent \( i \), say agent 1.

Multiply agent 1’s first set of equations in (4.5) by \( (dx^1)' \) to get \( (dx^1)' D^2 U^1(x^1) dx^1 - (dx^1)' Id \mu^1 = 0 \). From (4.5), we can substitute \( (dx^1)' I \) by \( (dz^1)' \tilde{W} \) to find:

\[
(dx^1)' D^2 U^1(x^1) dx^1 - (dz^1)' \tilde{W} d\mu^1 = 0
\]  (4.6)

Summing over agents, we get \( \sum_{i=1}^I (dx^i)' D^2 U^i(x^i) dx^i \), since \( \sum_{i=1}^I dz^i = 0 \).

Since there is at least one \( i \) such that \( dx^i \neq 0 \), from strict concavity we have \( \sum_{i=1}^I (dx^i)' D^2 U^i(x^i) dx^i < 0 \), a contradiction.

To finish this proof, we have to show that \( H^{-1}(0) \) is compact. Since \( H^{-1}(0) \) is a subset of \( \mathbb{R}^{dim \, \Xi} \), it is sufficient to show that every sequence in \( H^{-1}(0) \) has a convergent subsequence, and its limit point belongs to \( H^{-1}(0) \). Since \( \rho \in [0, 1] \), \( \rho_n \) is convergent. To check for the convergence of \( (x_n) \), note that \( (x_n) \) is bounded from below by zero. Then, \( -\sum_{i \neq 1} x_n^i \) is bounded from above, implying that the consumption of individual 1, which equals \( (1 - \rho_n) \sum_{i=1}^I \omega_n^i + \rho_n \sum_{i=1}^I x_{\delta n}^i - \sum_{i \neq 1} x_n^i \), is bounded from above. Assumption 1.ii guarantees closedness of the set \( \{ x_n : n \in \mathbb{N} \} \). Then, first order conditions \( \mu = DU(x) \) imply convergence of \( (\mu_n^i) \). From the first order conditions \( \mu' \tilde{W} = 0 \) we get convergence of prices. Finally, the remaining set of equations in the first order conditions (4.2), that is, \( \omega - x + \tilde{W} z = 0 \), guarantees convergence of \( z_n^i \). Continuity of \( F \) implies that the limit of all these sequences is in \( H^{-1}(0) \), concluding this proof.

Proof. Proof of Proposition 2

In the Constrained Financial Equilibrium there are \( \tau \) prices of assets arbitrarily set. For instance, conditions in (4.3) fix prices according to \( Rq = 0 \).

We want to show that perturbations on prices of restricted assets imply a different equilibrium allocation. Consider the equilibrium allocation as described in (4.3).
Suppose prices of constrained assets are perturbed and the equilibrium allocation remains the same. Then, strict concavity and the first set of equations in (4.3), namely \( DU(x) = \mu \), imply that the values of \( \mu \) are the same. Budget constraints imply that the amount of traded assets paying at time \( T \) must remain the same. For a given asset \( j \), define:

- \( s_j \) as the state where asset \( j \) is sold;
- \( V_j \) as the set of states where it promises some payoff;
- \( d_{n,j} \) as a dummy variable: \( d_{n,j} = 1 \) if there is a restriction \( z_j = z_n, j \neq n \) and \( d_{n,j} = 0 \) otherwise.

A typical element of \( \mu^W \) is:

\[
\sum_{j \in J} d_{n,j} \mu_{s_j} q_j + \mu_{s_n} q_n - \sum_{j \in J} \sum_{s \in V_j} d_{n,j} \mu_s - \sum_{s \in V_n} \mu_s
\]

Then, the equilibrium condition in (4.3) defines:

\[
\mu_{s_n} q_n = \sum_{j \in J} \sum_{s \in V_j} d_{n,j} \mu_s + \sum_{s \in V_n} \mu_s - \sum_{j \in J} d_{n,j} \mu_{s_j} q_j = 0
\]

Since the values of \( \mu_s \) are all fixed, arbitrary changes of prices \( q_j \), denoted as \( \Delta q_j \), imply:

\[
\Delta q_n = - \sum_{j \in J} \frac{\mu_{s_j}}{\mu_{s_n}} d_{n,j} \Delta q_j = 0 \tag{4.7}
\]

Suppose we can find common ratios \( d_{n,j} \mu_{s_j} / \mu_{s_n} \) for all agents. Then, we can substitute (4.7) at agents’ budget constraints. Then, each budget constraint at \( T - 1 \) where assets are sold is perturbed by arbitrary variations in asset prices. Because markets are incomplete, there will be at least one spot at which changes in prices cannot compensated by changes in the demand of an asset bought at \( T - 2 \). As shown by Geanakoplos and Mas-Colell (1989), generically individual assets demands span \( \mathbb{R}^r \), finishing this proof. \( \Box \)
References


