TEACHING AGHION AND HOWITT’S MODEL OF SCHUMPERETERIAN GROWTH TO GRADUATE STUDENTS: A DIAGRAMMATIC APPROACH*

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ABSTRACT

A mainstay of undergraduate courses on economic growth is an exposition of Robert Solow’s famous 1956 model. Solow’s famous diagram is an important vehicle for teaching this model in both undergraduate and early graduate courses on growth. A desire to extend Solow’s framework and to explain the determination of technical progress motivated the development of endogenous growth theory inaugurated by the seminal contribution of Paul Romer (1986). This eventually led to the Schumpeterian model of Aghion and Howit (1992) which is sufficiently complicated that it tends to be omitted from even introductory graduate courses. This paper presents a slightly simplified treatment of the Aghion-Howitt model that might be used to complement and extend treatments of growth theory in textbooks such as Jones (1998) and Romer (2012) for introductory graduate courses. The centerpiece of this treatment is a four-quadrant diagram that makes the key ideas of the model more accessible. The paper argues that diagrams and figures can be powerful pedagogical tools for teaching complex models, essentially rephrasing an old dictum as ‘one picture is worth a thousand equations’.

KEYWORDS: Endogenous growth, graduate teaching, innovation, creative destruction.

JEL classifications: A23, O31

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ISSN 1448-448X © 2012 Australasian Journal of Economics Education
1. INTRODUCTION
A mainstay of undergraduate courses on economic growth is an exposition of Robert Solow’s famous 1956 model.¹ In the Solow model, the long-run growth rate of output per worker is determined by technological progress but this variable is treated as being exogenous and is left un-modeled. A desire to extend this framework and to explain how this important driver of economic growth is determined, motivated the development of endogenous growth theory inaugurated by the seminal contribution of Paul Romer (1986).

This development occurred in three phases. In the first, so-called AK models hypothesized that high rates of growth depend upon thrift, some of which finances a higher rate of technological progress, resulting in higher growth (see Romer 1987; and Rebelo 1991). No explicit distinction was made, however, between technological progress and capital accumulation in these models. In the second phase, “innovation-based” models posited that innovation causes productivity growth by creating new varieties of intermediate goods. Here innovations do not necessarily generate better intermediate products, just more of them, and the increased use of these goods associated with their greater supply and variety leads to higher growth (see, for example, Romer 1990). In the third phase, innovation-based theory took a Schumpeterian approach. Aghion and Howitt (1992, 1998), for example, developed a model in which a version of Schumpeter’s process of creative destruction generates vertical innovations that drive the development of technological knowledge, increasing productivity, and fueling economic growth. Innovations in this model are the result of deliberate investment in research processes but because newly developed intermediate goods render existing ones obsolete, the expectation of future spending on research acts as a brake on current research spending, and firms must balance the costs and benefits of this spending.

The complexity of Aghion and Howitt’s original paper means that it tends to be excluded from undergraduate and early graduate treatments

¹ While Robert Solow is most often remembered for his contribution to the development of modern growth theory, it should be remembered that Trevor Swan (1956) independently published a model with virtually the same structure just a few months after Solow’s paper came out. See Spencer and Dimand (2010) for a brief discussion of the relationship between these contributions.
of economic growth. Jones (1998 pp.20-45; 88-113), for example, discusses the Solow model and outlines a simplified version of Romer’s (1990) model but provides only passing reference to the Aghion-Howitt model. Romer (2012 pp.6-45; 102-145), a text used in many introductory graduate courses, follows a similar pattern although more advanced mathematics are employed.

This paper presents a slightly simplified treatment of the Aghion-Howitt model that might be used to complement and extend treatments of growth theory in textbooks such as Jones (1998) and Romer (2012) for introductory graduate courses. The centerpiece of this treatment is a four-quadrant diagram that makes the key ideas of the model more accessible. The paper argues that diagrams and figures can be powerful pedagogical tools for teaching complex models, essentially rephrasing an old dictum as ‘one picture is worth a thousand equations’.

The paper is organized as follows. Section 2 provides the educational justification for developing the four-quadrant diagram. Section 3 outlines a simplified version of Aghion and Howitt’s (1992) model of Schumpeterian growth. Section 4 presents and explains the diagram, and Section 5 contains some brief concluding remarks.

2. USING DIAGRAMS TO TEACH ECONOMICS

Ramsden (2003, p.86) argues that effective learning, and teaching that facilitates such learning, is best conceptualized as a change in the way learners understand the world around them. The ‘world around them’ includes the concepts and methods central to the students’ field of learning. Such learning results when students focus on thoroughly engaging with course material and constitutes what Ramsden (2003, pp.46-47) calls a deep approach to learning which generates high-quality, well-structured learning outcomes. Surface approaches, on the other hand, encourage students to focus on activities such as merely reproducing facts and leads at best to an ability to retain unrelated details.

Some ways of addressing complex relationships do not, therefore, help students connect with the key ideas in an area of study, according to Ramsden’s perspective. Many students would have difficulty, for example, engaging with the exposition of a complex model in algebraic or mathematical form. For Ramsden (2003, p.88-89), a key feature of
effective teaching is the capacity to explain complex material plainly so that students can better access its underlying structure.

Economics instructors often try to achieve this goal by using curves and diagrams. Textbooks, in particular, make extensive use of diagrams to represent mathematical functions and economic relationships because some students have a stronger visual sense than a sense of numbers so that, for them, thinking about complex relationships is more readily facilitated by seeing them in spatial terms than it is by thinking about them abstractly using systems of equations. Solow’s famous diagram is a case in point. Being able to picture a region where saving exceeds breakeven investment so that the capital-labour ratio increases through time, and another region where the opposite is true, makes the idea of an equilibrium capital–labour ratio much more intuitively accessible (see, for example, Romer 2012, p.16). Even if it is important for students to understand the mathematics of a formal economic model, approaching the model by way of a diagrammatic structure first may pave the way for understanding the mathematics later.

Blaug & Lloyd (2010, p.5) point out that figures and diagrams have also been used in economics for high level analytical research functions such as discovering results and demonstrating proofs, as well as for pedagogy. Geometry is sometimes better suited than algebra to highlighting the essential features of a complex model characterized by a large number of variables and relationships. A well designed diagram can bring out the central features of a model, show interactions between relationships, and convey information contained in two or more curves that cannot be easily seen using equations or words. In economics ‘one picture is worth a thousand words’, but perhaps we should rephrase this old dictum as ‘one picture is worth a thousand equations’.

We use this pedagogical perspective to develop a geometric approach for illustrating the dynamic general equilibrium of the challenging Aghion and Howitt (1992) Schumpeterian growth model. We bring together in one diagram the key variables and functional relationships of the model so that we may keep track of complex interactions among sectors and show how these interact so as to be simultaneously in equilibrium. We hope this will be of use to graduate students and professors in thinking about how they might make a complex model more accessible to their students. The following section outlines the
precise structure of the Howitt and Aghion model we have in mind for exposition in diagrammatic form as explained in the section after that.

3. AGHION AND HOWITT’S MODEL
Aghion and Howitt’s (1992) model is technically complex and a first step towards generating the kind of diagram that will make the model more accessible to graduate students is to develop a simplified version of the model. A useful guide in this respect that we will follow is the approach taken by Jones (1998 pp.101-111) in outlining a simplified version of Romer’s (1990) model of endogenous growth. Jones’ approach makes a number of assumptions in addition to those made by Romer with the aim of significantly reducing the complexity of the model while still retaining the fundamental principles and results of Romer’s analysis. In this way students are able to follow the basic economic reasoning of the Romer model and to understand its basic results without having to struggle with the difficulty of Romer’s original paper. We take a parallel approach with respect to Aghion and Howitt (1992).

The fundamental idea of the Aghion and Howitt (1992) model is that entrepreneurs invest in research and development processes that enhance the productivity of intermediate (capital) goods used in the production of final goods. Such productivity-enhancing innovations drive the pace of technical progress and are thus a key source of economic growth. But new technologies produced by the research and development industry also render previous technologies obsolete so that those investing in this industry run the risk that the profitability of their innovations may be relatively short lived. Since they are profit maximisers, the amount of investment that underlies the discovery of innovations is the result of a business decision and is thus endogenous to the economy’s internal workings. We may thus think of research and development as generating a series of innovations that are reflected in the particular characteristics of the intermediate good. The $t^{th}$ innovation generates the $t^{th}$ intermediate good which replaces the $(t-1)^{st}$ intermediate good. The number $t$ therefore represents the number of innovations that have occurred to date.

Like Aghion and Howitt (1992, p.327) we will assume that the economy has four sectors: a final goods sector; an intermediate (capital) goods sector; a research and development sector; and a labour market.
Each sector is described in turn before the character of the economy’s general equilibrium is described.

(a) The Final Goods Sector
The economy produces a single final (consumption) good under conditions of perfect competition, using one input – a single intermediate good. Production of the final good is governed by a Cobb-Douglas-type production function of the following form:

\[ y = Ax^\alpha \]  

where: \( y \) is output of the consumption good; \( x \) is the amount of the intermediate good used to produce the final good; \( A \) is a productivity parameter that reflects the current quality of the intermediate good; and \( \alpha \) is a coefficient that lies between zero and one.

Recall that the final good is produced under perfect competition, using the intermediate product as the only input. If we take the price of the intermediate good as given, the profit maximizing choice of this input will be determined where the marginal revenue product of the intermediate good is equal to its unit price. Using the final good as numéraire, so that its unit price is equal to one, the following expression gives the condition for the profit maximizing choice of intermediate good use in the production of final goods:

\[ p_t = \frac{\partial y_t}{\partial x_t} = \alpha A_t x_t^{\alpha-1} \]  

where: \( p_t \) is the price of the \( t^{th} \) intermediate good. This expression may also be taken as the inverse demand curve for intermediate goods.

(b) The Intermediate Goods Sector
The intermediate good is produced using labour according to a simple one-for-one technology. That is, each unit of the intermediate good produced requires the input of one unit of labour. Thus \( x \) is also the amount of labour currently employed in production of the intermediate good:

\[ x_t = L_t \]  

where: \( L_t \) denotes the amount of labour used to produce the \( t^{th} \) intermediate good.
The intermediate sector is also assumed to be a monopoly industry. Firms that manufacture state-of-the-art intermediate goods enjoy monopoly power and earn positive profits by virtue of possessing patent rights over the production of particular intermediate goods which have been purchased from firms in the research and development sector. However since new technology is constantly being discovered in the research and development sector, patented technology is constantly being superseded in the intermediate goods sector. We will assume that firms in the intermediate goods sector purchase only one patent so that when that patent is superseded, they leave the industry to be replaced by a new firm that has purchased the latest patent technology. The identity of the monopoly firm producing in the intermediate goods sector is thus constantly changing.

Behaviour of the intermediate sector monopolist is also driven by maximizing profit which may be written as follows:

$$\pi_t = p_t x_t - w_t L_t$$

where: $\pi_t$ denotes profit from the $t^{th}$ innovation; and $w_t$ is the unit real wage paid to labour. Substituting for the price of intermediate goods from equation (2) and labour used to produce intermediate goods from equation (3), differentiating the resulting equation with respect to the choice of output for the intermediate good and setting equal to zero yields the first order condition for a maximum profit. If we also define the productivity-adjusted wage rate by the ratio of the real wage to the economy’s productivity level: $\omega_t \equiv w_t / A_t$, then the following equation is obtained after some rearranging:

$$\omega_t = \alpha^2 x_t^{2-\alpha}$$

We may refer to this equation as: $\omega_t = \tilde{\omega}(x_t)$. Solving equation (5) for the optimal level of intermediate good production yields:

$$x_t^* = \left(\frac{\alpha^2}{\omega_t}\right)^{\frac{1}{1-\alpha}}$$

and this expression may also be referred to as: $x_t = \tilde{x}(\omega_t)$. This function is also the demand for labour in the intermediate goods industry and it
implies that this demand is a decreasing function of the productivity-adjusted wage rate, $\omega_t$.

Profit in the intermediate goods sector can be expressed as a function of the quality of the intermediate product, $A_t$, and the productivity adjusted wage. Assume that the intermediate goods firm practices markup pricing, with the price-elasticity of demand facing the firm being constant. This elasticity, $\varepsilon$, is $1/(\alpha - 1)$ and the monopolist’s price is given by $p_t = w_t/(1 + \varepsilon^2) = w_t/\alpha = A_t(1/\alpha)\omega_t$. Now, substituting for $p_t$ in equation (4) and collecting terms yields a new expression for the profit function:

$$
\pi_t = A_t \left( \frac{1}{\alpha} - 1 \right) \omega_t x_t, \quad (7)
$$

We may also substitute for $x_t$ from equation (6) to get:

$$
\pi_t = A_t \left( 1 - \frac{1}{\alpha} \right) \left( \frac{\alpha^2}{\omega_t^\alpha} \right)^{\frac{1}{1-\alpha}}, \quad (8)
$$

which we may abbreviate as $\pi_t = A_t \bar{\pi}(\omega_t)$ and where $\pi_t$ is a decreasing function of $\omega_t$.

(c) The Research and Development Sector

Improvements in the productivity, and thus in the quality, of the intermediate good are the result of technological innovations, and the role of research and development firms is to make investments aimed at creating these innovations. Aghion & Howitt (1992, p.329) model innovations as occurring with a Poisson arrival rate of $\lambda \cdot \phi(z,s)$ where $z$ and $s$ are types of labour used in the research and development process. We simplify this arrival rate to $\lambda n$, where $\lambda$ is a probability parameter and $n$ is the amount of labour used in the research and development industry. This means that increasing the amount of labour used in research and

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2 A different interpretation of result (5) is that labour in the intermediate goods industry is paid its (productivity-adjusted) marginal-revenue product. It is not the current wage rate that really matters for rational decision making in the intermediate sector but the wage rate relative to the economy’s productivity.

3 Alternatively substitute equation (6) into equation (2) to give $p_t = (1/\alpha)w_t$ or $p_t = A_t(1/\alpha)\omega_t$. 
development can increase the rate at which innovations arrive and $\lambda$ is a kind of measure of the industry’s own productivity. When an innovation does occur, it enhances the productivity of intermediate goods in the production of final goods by a factor of $\gamma > 1$. The productivity of the new “state-of-the-art” intermediate good is thus given by:

$$A_t = \gamma A_{t-1}$$

(9)

where $t$ again indexes the $t^{th}$ innovation rather than time.$^4$

The research and development sector is assumed to be competitive. This competition can be likened to the race for a patent. Because the market delivers a profit to the monopoly situation in the production of intermediate goods, there is competition for the patents that create this monopoly situation. Research and development firms try to be the next firm to innovate, obtain a patent and then sell the rights to produce the intermediate good under this patent. Whenever a new innovation occurs, the rights conferred under the new patent make those under the previous one obsolete, and monopoly profits associated with producing the old intermediate good evaporate. A successful innovator today therefore becomes the monopolist tomorrow, or in our set up, sells the patent to the firm that becomes the monopolist tomorrow in the production of intermediate goods.

To determine the amount of research and development that should be undertaken by firms in this sector will equate the marginal cost of a unit of research labour with the expected marginal benefit. Marginal cost is simply the wage, $w_t$. The expected marginal benefit arises from raising the probability of innovation by $\lambda$ for each additional worker employed. If we denote $V_{t+1}$ as the value of the next patent, research firms should thus hire labour for the research and development process up to the point where the following condition holds:

$$w_t = \lambda V_{t+1}$$

(10)

If the value of the new patent is invested for a unit time interval (say one day) at the risk free rate, $r$, the return will be $rV_{t+1}$. If this amount is

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$^4$ A time period can be thought of in the model as the interval between two successive innovations, the symbol $\Delta_t$ being used to denote the interval starting with the $t^{th}$ innovation and ending just before the $(t + 1)^{st}$ innovation.
alternatively used to purchase the patent, the return in the unit time interval period will be the monopoly profits associated with the $t+1^{st}$ innovation plus the expected rate of capital appreciation (or loss) that results from the change in value of the patent.\(^5\) This can be written as $\pi_{t+1} + dV^e_{t+1}$. The second part of this expression is related to the probability of a new innovation being discovered that makes the patent worthless. This probability is $\lambda n_{t+1}$ and generates an expected capital loss of $[\lambda n_{t+1} \cdot 0 + (1 - \lambda n_{t+1})V_{t+1}] - V_{t+1}$. Expanding this expression allows the expression for the return from investing in the patent to be written as $\pi_{t+1} - \lambda n_{t+1}V_{t+1}$. In equilibrium, the returns from investing in the risk free rate and the patent must be the same. This equilibrium, or no arbitrage, condition may be written as:

$$rV_{t+1} = \pi_{t+1} - \lambda n_{t+1}V_{t+1}. \quad (11)$$

Solving this expression for $V_{t+1}$ yields:

$$V_{t+1} = \frac{\pi_{t+1}}{r + \lambda n_{t+1}} \quad \text{(12)}$$

where the numerator reflects monopoly profit and the denominator can be seen as the “effective” interest rate, that is, the discount rate adjusted by the obsolescence rate of the knowledge underlying the monopoly profit. This equation, therefore, gives the value of the patent associated with the $(t + 1)^{st}$ innovation.

In the previous sub-section we represented monopoly profit in the intermediate goods sector associated with $t^{th}$ innovation as a linear function of the productivity level $A_t$: $\pi_{t+1} = A_t \gamma \bar{\pi}(\omega_{t+1})$. Substituting for $\pi_{t+1}$ in the research arbitrage equation (10) above, the factor $\gamma$ shows on the right-hand side of the equation. Then, dividing through by $A_t$, the research arbitrage equation can now be re-expressed as:

$$\omega_t = \frac{\lambda \gamma \bar{\pi}(\omega_{t+1})}{r + \lambda n_{t+1}} \quad \text{(13)}$$

Thus the allocation of labour between research and the production of the intermediate good is a function of the productivity-adjusted wage rate $\omega_t$.

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\(^5\) Jones (1998, pp.106-107) has a good description of this method.
The concept of arbitrage also proves to be useful in determining what fraction of the population works in each sector.

The equilibrium level of labour employed in the research sector is neither determined by the current wage rate, nor affected by the actual change of the productivity parameter stemming from an innovation. The first-order condition for the maximization problem of research firms suggests that the marginal benefit of an additional unit of research labour is higher, the higher the productivity increase in the economy. But this advantage is just offset if the marginal cost (the wage rate) of an additional unit of labour is also higher because of the higher productivity. Thus the optimal solution remains the same as before any increase in productivity.

(d) The Labour Market

The labour market is also assumed to be competitive and frictionless. We assume that there are \( N \) individuals in the labour market, each of whom has one unit of labour to supply inelastically (that is, no matter what the wage rate). Equation (14) is a labour market clearing equation, which states that for each innovation total labour supply \( N \) is equal to employment in the intermediate goods industry, \( x_t \), plus employment in the research and development industry, \( n_t \). Thus the economy faces a resource constraint defined in labour units.

\[
N = x_t + n_t
\]  

(14)

Substituting for \( x_t \) from equation (6) this becomes:

\[
N = \tilde{x}(\omega_t) + n_t
\]  

(15)

In equilibrium, all workers must be paid the same wage. The price of labour will, therefore, adjust so equation (15) is satisfied and the productivity-adjusted wage rate \( \omega_t \) is a function of the labour market clearing condition.

(e) Equilibrium Research and Growth

General equilibrium in the aggregate economy requires that both conditions (13) and (15) are simultaneously satisfied. We are now in a position to find the equilibrium value of research employment. By
substituting for \( \omega_t \) in the research arbitrage equation (13) using the marginal-revenue function (5), dividing through by \( \lambda \), and using the labour market clearing equation (15) to substitute for \( x_t \), we may obtain the following:

\[
\frac{\alpha^2 (N - n_t)^{a-1}}{\lambda} = \frac{\gamma \alpha (1 - \alpha)(N - n_{t+1})^a}{r + \lambda n_{t+1}}
\]  \hspace{1cm} (16)

The left-hand side of equation (16) is an increasing function of \( n_t \) and can be interpreted as the “marginal cost of research” \( c(n_t) \). The right-hand side is a decreasing function of \( n_{t+1} \) and can be interpreted as the “marginal benefit of research” \( b(n_{t+1}) \). Thus equilibrium in the economy requires \( c(n_t) = b(n_{t+1}) \).

Individual researchers, being small relative to the economy as a whole, take the wage rate as given. They do not recognize that the wage increases as more labour enters the research sector. In other words, the many small research firms are price takers in the labour market. The research arbitrage equation (13) above illustrates this assumption. In the economy as a whole, however, the wage rate earned by labour in the research sector during the development of a particular innovation is an increasing function of the amount of research during the period of time over which that innovation is being developed. While the wage will change by only a negligible amount in response to the research efforts of a single researcher, it clearly varies with aggregate research effort. In the model, the productivity-adjusted wage rate \( \omega_t \) is increasing in \( n_t \). To see this, note from equation (5) that the marginal-revenue product of labour in the manufacturing sector is decreasing in manufacturing employment \( x_t \), and from equation (15) that \( x_t \) equals the residual supply of manufacturing labour \( N - n_t \). The implication of this result of the model is that the marginal cost of research \( c(n_t) \) is increasing and the marginal benefit of research \( b(n_{t+1}) \) is decreasing.

Equilibrium condition (16) determines the amount of current research as a decreasing function of the amount of expected future research: \( n_t = \psi(n_{t+1}) \). It is demonstrated in the Appendix that the functional relationship \( \psi \) in the economy is given by:
The fact that the functional relationship $\psi$ between research employment during the development of two consecutive innovations is strictly decreasing suggests that, in equilibrium, the allocation of labour between research and manufacturing is likely to change over time. However, rather than taking a general approach to finding the equilibrium, we will look for an equilibrium where the amount of labour employed in research is constant.

In a steady-state (or stationary) equilibrium the allocation of labour between research and manufacturing remains constant. In a steady state the expected growth rate of final output (or consumption good) in the economy is also constant over time. The economy has a unique steady-state equilibrium for a given set of parameter values. To determine the economy’s growth rate, we need to determine the equilibrium value of research employment $\hat{n}$. Using the fact that the amount of labour employed in research remains constant in a steady-state, it is simplest to leave out the subscript $t$.

Condition (17) will pin down the equilibrium value of research employment. That is, solving $n = \psi(n)$ for the steady-state amount of research labour yields the expression:

$$n_t = N - \left( \frac{\lambda \gamma \left(1 - \frac{\alpha}{\alpha} (N - n_{t+1})^\alpha \right)^{\frac{1}{\alpha - 1}}}{r + \lambda n_{t+1}} \right)$$

(17)

One can show that this level of research will produce an economy’s average growth rate equal to:

$$g = \lambda \hat{n} \ln \gamma$$

(19)

---

6 Proof of this solution is shown in the Appendix.
The equilibrium expected rate of growth in steady-state is determined by the characteristics of the economic environment as described by the parameters $\lambda$, $\gamma$, $\alpha$, $r$, and $N$ of the model. In particular, parameters $\lambda$ and $\gamma$ affect growth at least in part by determining the amount of labour employed in research according to result (18).

The economy’s expected growth rate (19) equals the probability of innovation $\lambda \hat{n}$ times the size of innovation $\ln \gamma$. How does final-good production in the economy evolve over time in a steady-state? Economic growth results from innovations that raise the productivity parameter $A_t$. The economy’s growth rate is the proportional growth rate of the productivity parameter $A_t$. To see this, note first that final good output is proportional to the productivity of the intermediate good according to equations (1) and (18), the latter of which implies the equilibrium amount of manufacturing labour. That is, $y_t = A_t(N - \hat{n})^\alpha$. This result, together with the fact that $A_{t+1} = \gamma A_t$, implies that $y_{t+1} = \gamma y_t$. Now suppose that $t$ innovations have occurred up to the present date $\tau$. Consider the unit-time interval between $\tau$ and $\tau + 1$. Then, with probability $\lambda \hat{n}$, research firms will succeed in discovering innovation number $t + 1$, and consequently $g(\tau) = \ln y(\tau + 1) - \ln y(\tau) = \ln A(\tau + 1) - \ln A(\tau)$. With probability $1 - \lambda \hat{n}$, research firms will fail to innovate, and consequently $g(\tau) = 0$. Thus, the economy’s expected growth rate between $\tau$ and $\tau + 1$ is equal to $\lambda \hat{n} \ln \gamma + (1 - \lambda \hat{n}) \cdot 0 = \lambda \hat{n} \ln \gamma$.

This outlines the complete theoretical structure of the simplified Aghion and Howitt Model. We can now outline a diagrammatic representation of this model which we think will make its interpretation and learning by students easier.

4. A DIAGRAM TO ILLUSTRATE THE MODEL

The model outlined in the previous section can be viewed as being made up four key variables and four functional relationships. The four key variables are: the volume of final goods produced, $y_t$; the volume of intermediate goods produced, $x_t$; the productivity-adjusted wage, $\omega_t$; and the amount of labour currently employed in the research and development sector, $n_t$. A fifth variable is, however, also relevant which is the amount of labour employed in research and development in the next period, $n_{t+1}$. Earlier in the paper we suggested that a time period can be thought of as the interval between two successive innovations, the
symbol $\Delta_t$ being used to denote the interval starting with the $t^{th}$ innovation and ending just before the $(t + 1)^{st}$ innovation. It is important to keep in mind this conception of time in the model. The key variables are determined jointly by the functional relationships shown in expressions (1), (6), (13), and (15) which are reproduced below.

\[
y = Ax^\alpha
\]

\[
x_t = \left(\frac{\alpha^2}{\omega_t}\right)^{\frac{1}{1-\alpha}}
\]

\[
\omega_t = \lambda \frac{\gamma(\omega_{t+1})}{r + \lambda n_{t+1}}
\]

\[
n_t = N - \tilde{x}(\omega_t)
\]

In developing our four quadrant diagram, we will put aside equation (1) and output in the final goods sector (which will ultimately be the variable of interest) and return to these aspects of the model later. For the moment, our attention will focus on equations (6), (13) and (15) and the four endogenous variables within these equations: $x_t$, $\omega_t$, $n_t$ and $n_{t+1}$. Equation (6) represents the demand for labour in the intermediate goods sector as a function of the productivity-adjusted wage. This relationship is shown in the third quadrant (bottom right) of Figure 1 where employment $x_t$ is a decreasing function of the productivity-adjusted wage rate $\omega_t$. Equation (15) is the equilibrium condition for the overall labour market and this is shown in quadrant IV of Figure 1 as a straight line with slope minus one in $(x_t, n_t)$ space. Because the labour market is competitive, all transactions in the labour market occur at the same market price. With aggregate labour supply equal to $N$, the productivity-adjusted wage rate $\omega_t$ will, therefore, be determined by equation (15).

Equation (13) represents the inverse relationship between $n_{t+1}$ and $\omega_t$ and is depicted by curve A in the second quadrant of Figure 1. This research arbitrage condition indicates the incentive for firms in the research and development sector to invent a new intermediate good. Innovations arrive randomly but the probability of innovations depends positively on the amount of research labour employed $n_t$. Research firms will thus choose the amount of labour so that the marginal benefit of
employing labour, which is decreasing in $n_t$ under the general case because the incremental probability of innovation is a decreasing function of $n_t$, is equal to the marginal cost which is increasing in $n_t$ because the productivity-adjusted wage rate is increasing in $n_t$. Hence, the arbitrage equation implies the equilibrium research amount $n_t$. The research arbitrage curve (A) is thus decreasing in the future amount of labour employed because the value of making the next innovation is decreasing in the future amount of research. The value of making the next innovation is decreasing in the probability of innovation $\lambda n_{t+1}$ (under the linear research technology case), where $\lambda$ is a parameter indicating the productivity of the research sector. Consequently the current wage decreases as well. The research arbitrage condition (A) governs the dynamics of the economy over its successive innovations. The optimal investments in innovations are determined by this condition and govern the dynamics of the economy. Observe that the negative slope of the curve corresponding to (A) reflects the sum of two elements, the influence of a creative destruction effect, and the impact of two general
equilibrium effects, with implications for the slope of the functional relationship between research in two successive periods in the first quadrant. The general equilibrium effects of wages on profits created by current research and of the level of manufacturing employment on wages are added to the following analysis.

A higher level of research \( n_{t+1} \) tomorrow implies two things. The first is a higher rate of creative destruction next period, that is a higher probability of innovation shortening the expected lifetime of the monopoly to be enjoyed by the next innovator. The second is higher future wages \( \omega_{t+1} \), lessening the stream of profits to be appropriated by the next innovator. The second effect is indicated by pairs of boldfaced arrows (\( \rightarrow \)) in \((x_t, n_t)\) and \((x_t, \omega_t)\) spaces, with current period subscript \( t \) in the axes of both spaces now being replaced by future period subscript \( t + 1 \). This will lower the discounted expected payoff of the \((t + 1)^{th}\) innovation and, hence, will discourage the amount of research today. Note that the second effect above is the general equilibrium effect of future research on the profits created by current research referred to above. In turn, a lower expected value of an hour in research will imply a lower current wage \( \omega_t \), hence explaining the negative slope of the curve corresponding to (A).

Equations (6), (13) and (15) thus constitute the foundation on which Figure 1 is constructed. But it will be observed that this subsystem of three equations contains four endogenous variables. The structure of this system, however implies a relationship between \( n_t \) and \( n_{t+1} \) which is essentially a forward-looking difference equation of the form \( n_t = \psi(n_{t+1}) \), and this equation is important for the dynamic behavior of the economy and for its convergence to an equilibrium growth path.

In the previous section of the paper we derived this implicit relation in the form of expression (17) which is also reproduced below.

\[
n_t = N - \left( \frac{\lambda \gamma}{\alpha} \left( N - n_{t+1} \right)^\alpha \right)^{1/(\alpha - 1)} - \frac{1}{r + \lambda n_{t+1}}
\]

(17)

But we can also obtain this relation diagrammatically in Figure 1. From the representation of relationships in the second through fourth
In three quadrants, we can derive the $\psi(n_{t+1})$ curve in the first quadrant. We begin with a particular value for the level of labour employed in future research and development, $n_{t+1,0}$, in quadrant II, and trace the corresponding values of the variable sequence $\omega_t$, $x_t$, $n_t$ through the relationships in quadrants II, III, and IV. This is shown by the rectangle $B_3B_2B_1B$ and ultimately gives a value of $n_{t,0}$ for labour currently employed in research and development in panel IV. We can plot the resulting $(n_t, n_{t+1})$ combination as point B in quadrant I. If we continue to choose values for $n_{t+1}$ and find their corresponding $n_t$ values, and plot the resulting combinations in quadrant I, we will define the $\psi(n_{t+1})$ curve in that quadrant.

This is the curve of the relationship in equation (17) and its negative relationship between labour employed in current and future research tells us that a higher amount of future research will discourage current research. A higher amount of future research will decrease the marginal benefit of a unit of research labour (through the influence of a creative destruction effect and the impact of a general equilibrium effect), which will imply a lower current wage (in order to reestablish the equality between the marginal cost of research and the marginal benefit of research taken from the research arbitrage equation). The graphical depiction of the response of current wages to changes in future research could be shown as downward movements along the curve (A) in the second quadrant of Figure 1 above. Given that workers contribute less than before to productivity growth and profits if employed in research rather than in manufacturing, the demand for labour will tend to increase in manufacturing, inducing a greater fraction of labour to move away from research. This in turn will reduce the value of the marginal product of labour in manufacturing and will push down current wages. Those employment changes could be represented by downward movements along the demand manufacturing labour curve $\tilde{x}(\omega_t)$ and the residual supply research labour curve $n_t = N - x_t$ (taken from the labour market-clearing equation (L)), respectively in the third and fourth quadrants.

Determination of equilibrium in Figure 1 now hinges on a solution to the forward-looking difference equation $n_t = \psi(n_{t+1})$ represented by the downward sloping curve in quadrant I. If we begin arbitrarily at point B, the current level of research is $n_0$. Moving horizontally to the right towards the $45^\circ$ line in the first quadrant and then vertically downward,
we register \( n_j \), which is the horizontal coordinate of point C on the \( \psi(n_{t+1}) \) curve and the value of \( n \) that we could expect to be chosen next period. This will eventually form a sequence \( \{n_0, n_1, n_2, \ldots\} \) constructed from the clockwise spiral starting at \( n_0 \) converging to the point where the \( \psi(n_{t+1}) \) curve intersects with the 45º-line. The resulting choice of labour allocated to research and development is thus stable through time and is the same value as that given by expression (18) in the previous section of the paper.

Economically, we can think about this result in terms of agent maximization and perfect foresight. Aghion and Howitt’s version of the Schumpeterian model abstracts from elements such as uncertainty about future research efforts and uses these modern tools of agent maximization and perfect foresight. Under perfect foresight, research firms make no mistakes while choosing the amount of labour employed in research each period and correctly anticipate or foresee any increase or decrease in research next period. In this manner the model does not rely on agents’ expectations that are repeatedly and systematically fooled along the solution path of the economy. Research firms form expectations about the path of research efforts and these expectations are fulfilled by the chosen amounts of labour employed in innovation. In this sense any perfect foresight equilibrium is a self-fulfilling equilibrium. A perfect foresight equilibrium (PFE) is defined as a sequence \( \{n_t\}' \) satisfying \( n_t = \psi(n_{t+1}) \) for all \( t \geq 0 \). In quadrant I of Figure 1, the sequence \( \{n_0, n_1, n_2, \ldots\} \) that we constructed above from the clockwise spiral starting at \( n_0 \) constitutes a PFE.

Now that the equilibrium values for \( n_t \) and \( n_{t+1} \) have been determined, we may substitute the value for \( n_{t+1} \) into the function in quadrant II to find the equilibrium value for \( \omega_t \), and the value for \( n_t \) into the function in quadrant IV to find the equilibrium value for \( x_t \), which will be consistent with the values for these variables from the function in quadrant III. Economically both of these variables will be determined simultaneously by the dynamic process which establishes the values of \( n_t \) and \( n_{t+1} \) in the first place. We may then substitute the value for \( x_t \), into equation (1) for the output of final commodities and determine the growth rate by the process outlined at the end Section 3.

While we have presented the development of our diagram by first outlining the mathematics of the simplified Aghion and Howitt model
and then describing the diagram, we have done this for the sake of exposition to professors. Pedagogically, we would approach exposition of the model very differently. For presentation to graduate students, we would outline the assumptions of the model and explain the economics of equations (1), (6), (13) and (15) which we think would be relatively straightforward. We would then plot each of these functions in quadrants II through IV without anything appearing in quadrant I. We would then derive the function in quadrant I diagrammatically, insert the 45º line and explain the intuition of the convergent dynamic process that establishes the equilibrium values for $n_t$ and $n_{t+1}$. We would then back out the equilibrium values for $\omega_t$ and $n_t$ as outlined above and describe the implications for growth using equation (1) and the brief discussion of growth at the end of Section 3, largely as one does in the Solow model once the equilibrium value for the capital-labour ratio is determined. Only then would we engage in the full mathematical treatment of the model and review the core economics of what the model represents.

5. CONCLUSION

This paper has argued that the geometric method is very effective in getting key concepts across to students and providing clear explanations of complex ideas in economic modeling. It can thus be a powerful and flexible pedagogical tool even for teaching graduate students. We have used these principles to develop a diagram which illustrates the key features of Aghion and Howitt’s (1992) model of endogenous growth through creative destruction. We outline a simplified version of the Aghion and Howitt model, trace through the mathematics of the model, present out four quadrant diagram and then explain briefly how this material would be presented to graduate students. Now we can only hope that we have also had the ability to make the Aghion and Howitt material genuinely interesting, so that students find it a pleasure to learn.

REFERENCES


APPENDIX

In this appendix we derive results (17) and (18).

Proof of result (17):
Recall the economy’s equilibrium condition (16):

$$\frac{\alpha^2(N-n_t)^{a-1}}{\lambda} = \frac{\gamma\alpha(1-\alpha)(N-n_{t+1})^a}{r + \lambda n_{t+1}}.$$

Then, multiplying through by $\lambda/\alpha^2$, we obtain:

$$(N-n_t)^{a-1} = \frac{\lambda\gamma\alpha^{-1}(1-\alpha)(N-n_{t+1})^a}{r + \lambda n_{t+1}}.$$
Now raising both sides of this equation to the power $1/(\alpha - 1)$, we obtain:

$$N - n_t = \left( \frac{\lambda_0 \alpha^{-1} (1 - \alpha)(N - n_{t+1})^\alpha}{r + \lambda n_{t+1}} \right)^{1/(\alpha - 1)}$$

and therefore result (17) holds:

$$n_t = N - \left( \frac{\lambda_0 \alpha^{-1} (N - n_{t+1})^\alpha}{r + \lambda n_{t+1}} \right)^{1/(\alpha - 1)}.$$

**Proof of result (18):**

Now, the starting point is equation (17). Specifically, we solve $n = \psi(n)$ for the steady-state amount of research labour. From (17), after successive algebraic manipulations, we obtain:

$$-(N - n) = \left( \frac{\lambda_0 \alpha^{-1} (N - n)^\alpha}{r + \lambda n} \right)^{1/(\alpha - 1)}$$

or:

$$N - n = \left( \frac{\lambda_0 \alpha^{-1} (N - n)^\alpha}{r + \lambda n} \right)^{1/(\alpha - 1)}$$

followed by:

$$(N - n)^{\alpha - 1} = \frac{\lambda_0 \alpha^{-1} (N - n)^\alpha}{r + \lambda n}$$

and then by $r + \lambda n = \lambda_0 \alpha^{-1} (N - n)$. Now, collecting terms, we obtain

$$n(\lambda + \alpha^{-1} \lambda_0 \alpha) = \lambda_0 \alpha^{-1} (N - n) N - r.$$ Result (18) immediately follows from here:

$$\hat{n} = \frac{\lambda_0 \alpha^{-1} N - r}{\lambda \left( 1 + \gamma \alpha^{-1} \right)}.$$